Mikls Schweitzer 1952

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by Albanian Eagle

1 Find all convex polyhedra which have no diagonals (that is, for which every segment connecting two vertices lies on the boundary of the polyhedron).

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2 Is it possible to find three conics in the plane such that any straight line in the plane intersects at least two of the conics and through any point of the plane pass tangents to at least two of them?

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3 Prove:If $a=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ is a perfect number, then $2<\prod_{i=1}^n\frac{p_i}{p_i-1}<4$;
if moreover, a is odd, then the upper bound 4 may be reduced to $2\sqrt[3]{2}$.

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4 Let K be a finite field of p elements, where p is a prime. For every polynomial $f(x)=\sum_{i=0}^n a_ix^i$ ($\in K[x]$)
put $\overline{f(x)}=\sum_{i=0}^n a_ix^{p^i}$.
Prove that for any pair of polynomials $f(x),g(x)\in K[x]$, $\overline{f(x)}|\overline{g(x)}$ if and only if $f(x)|g(x)$.

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5 Let G be anon-commutative group. Consider all the one-to-one mappings $a\to a'$ of G onto itself such that $(ab)'=b'a'$ (i.e. the anti-automorphisms of G). Prove that this mappings together with the automorphisms of G constitute a group which contains the group of the automorphisms of G as direct factor.

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6 Let $2n$ distinct points on a circle be given. Arrange them into disjoint pairs in an arbitrary way and join the couples by chords. Determine the probability that no two of these n chords intersect. (All possible arrangement into pairs are supposed to have the same probability.)

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7 A point P is performing a random walk on the X-axis. At the instant $t=0$, P is at a point x_0 ($|x_0|\le N$, where x_0 and N denote integers, $N>0$). If at an instant t (t being a nonnegative integer), P is at a point of x integer abscissa and $|x|<N$, then by the instant $t+1$ it reaches either the point $x+1$ or the point $x-1$, each with probability $\frac{1}{2}$. If at the instant t, P is at the point $x=N$ [$x=-N$], then by the instant $t+1$ it is certain to reach the point $N-1$ [$-N+1$]. Denote by $P_k(t)$ the probability of P being at $x=k$ at instant t (k is an integer). Find $\lim_{t\to\infty}P_k(2t)$ and $\lim_{t\to\infty}P_k(2t+1)$ for every fixed k.

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8 For which values of z does the series $\sum_{n=1}^{\infty}c_1c_2\cdots c_nz^n$
converge, provided that $c_k>0$ and $\sum_{k=1}^{\infty}\frac{c_k}{k}<\infty$?

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9 Let C denote the set of functions $f(x)$, integrable (according to either Riemann or Lebesgue) on (a,b), with $0\le f(x)\le 1$. An element $\phi(x)\in C$ is said to be an "extreme point" of C if it can

not be represented as the arithmetical mean of two different elements of C. Find the extreme points of C and the functions $f(x) \in C$ which can be obtained as "weak limits" of extreme points $\phi_n(x)$ of C.
(The latter means that $\lim_{n\to\infty} \int_a^b \phi_n(x)h(x)\,dx = \int_a^b f(x)h(x)\,dx$
holds for every integrable function $h(x)$.)

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10 Let n be a positive integer. Prove that, for $0 < x < \frac{\pi}{n+1}$, $\sin x - \frac{\sin 2x}{2} + \cdots + (-1)^{n+1}\frac{\sin nx}{n} - \frac{x}{2}$ is positive if n is odd and negative if n is even.
