Mikls Schweitzer 1962

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by ehsan2004

1 Let f and g be polynomials with rational coefficients, and let F and G denote the sets of values of f and g at rational numbers. Prove that $F=G$ holds if and only if $f(x)=g(ax+b)$ for some suitable rational numbers $a\neq 0$ and b.

E. Fried

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2 Determine the roots of unity in the field of p-adic numbers.

L. Fuchs

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3 Let A and B be two Abelian groups, and define the sum of two homomorphisms η and χ from A to B by

$$a(\eta+\chi)=a\eta+a\chi \text{ for all } \ a\in A\ .$$

With this addition, the set of homomorphisms from A to B forms an Abelian group H. Suppose now that A is a p-group (p a prime number). Prove that in this case H becomes a topological group under the topology defined by taking the subgroups p^kH $(k=1,2,...)$ as a neighborhood base of 0. Prove that H is complete in this topology and that every connected component of H consists of a single element. When is H compact in this topology? [L. Fuchs]

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4 Show that

$$\prod_{1\leq x<y\leq \frac{p-1}{2}}(x^2+y^2)\equiv (-1)^{\lfloor \frac{p+1}{8}\rfloor} \ (\textbf{mod}\ p\ )$$

for every prime $p\equiv 3$ (**mod** 4). [J. Suranyi]

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5 Let f be a finite real function of one variable. Let $\overline{D}f$ and $\underline{D}f$ be its upper and lower derivatives, respectively, that is,

$$\overline{D}f=\limsup_{h,k\to 0_{h,k\geq 0_{h+k>0}}}\frac{f(x+h)-f(x-k)}{h+k}$$

,

$$\underline{D}f=\liminf_{h,k\to 0_{h,k\geq 0_{h+k>0}}}\frac{f(x+h)-f(x-k)}{h+k}.$$

Show that $\overline{D}f$ and $\underline{D}f$ are Borel-measurable functions. [A. Csaszar]

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6 Let E be a bounded subset of the real line, and let Ω be a system of (non degenerate) closed intervals such that for each $x \in E$ there exists an $I \in \Omega$ with left endpoint x. Show that for every $\varepsilon > 0$ there exists a finite number of pairwise non overlapping intervals belonging to Ω that cover E with the exception of a subset of outer measure less than ε. [J. Czipszer]

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7 Prove that the function

$$f(\nu) = \int_1^{\frac{1}{\nu}} \frac{dx}{\sqrt{(x^2-1)(1-\nu^2x^2)}}$$

(where the positive value of the square root is taken) is monotonically decreasing in the interval $0 < \nu < 1$. [P. Turan]

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8 Denote by $M(r, f)$ the maximum modulus on the circle $|z| = r$ of the transcendent entire function $f(z)$, and by $M_n(r, f)$ that of the nth partial sum of the power series of $f(z)$. Prove that the existence of an entire function $f_0(z)$ and a corresponding sequence of positive numbers $r_1 < r_2 < ... \to +\infty$ such that

$$\limsup_{n\to\infty} \frac{M_n(r_n, f_0)}{M(r_n, f_0)} = +\infty$$

[P. Turan]

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9 Find the minimum possible sum of lengths of edges of a prism all of whose edges are tangent of a unit sphere. [Muller-Pfeiffer].

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10 From a given triangle of unit area, we choose two points independetly with uniform distribution. The straight line connecting these points divides the triangle. with probability one, into a triangle and a quadrilateral. Calculate the expected values of the areas of these two regions. [A. Renyi]
