Mikls Schweitzer 1963

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by ehsan2004

1 Show that the perimeter of an arbitrary planar section of a tetrahedron is less than the perimeter of one of the faces of the tetrahedron. [Gy. Hajos]

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2 Show that the center of gravity of a convex region in the plane halves at least three chords of the region. [Gy. Hajos]

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3 Let $R=R_1 \oplus R_2$ be the direct sum of the rings R_1 and R_2, and let N_2 be the annihilator ideal of R_2 (in R_2). Prove that R_1 will be an ideal in every ring $\widetilde{R}$ containing R as an ideal if and only if the only homomorphism from R_1 to N_2 is the zero homomorphism. [Gy. Hajos]

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4 Call a polynomial positive reducible if it can be written as a product of two nonconstant polynomials with positive real coefficients. Let $f(x)$ be a polynomial with $f(0)\neq 0$ such that $f(x^n)$ is positive reducible for some natural number n. Prove that $f(x)$ itself is positive reducible. [L. Redei]

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5 Let H be a set of real numbers that does not consist of 0 alone and is closed under addition. Further, let $f(x)$ be a
real-valued function defined on H and satisfying the following conditions:

$$f(x) \leq f(y) \text{ if } x \leq y$$

and

$$f(x+y)=f(x)+f(y) \;(x,y \in H)\;.$$

Prove that $f(x)=cx$ on H, where c is a nonnegative number. [M. Hosszu, R. Borges]

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6 Show that if $f(x)$ is a real-valued, continuous function on the half-line $0 \leq x < \infty$, and

$$\int_0^{\infty} f^2(x)dx < \infty$$

then the function

$$g(x)=f(x)-2e^{-x}\int_0^x e^t f(t)dt$$

satisfies

$$\int_0^{\infty} g^2(x)dx=\int_0^{\infty} f^2(x)dx.$$

[B. Szokefalvi-Nagy]

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7 Prove that for every convex function $f(x)$ defined on the interval $-1 \le x \le 1$ and having absolute value at most 1,
there is a linear function $h(x)$ such that

$$\int_{-1}^{1} |f(x)-h(x)|dx \le 4-\sqrt{8}.$$

[L. Fejes-Toth]

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8 Let the Fourier series

$$\frac{a_0}{2}+\sum_{k\ge 1}(a_k \cos kx + b_k \sin kx)$$

of a function $f(x)$ be
absolutely convergent, and let

$$a_k^2+b_k^2 \ge a_{k+1}^2+b_{k+1}^2 \; (k=1,2,...) \; .$$

Show that

$$\frac{1}{h}\int_0^{2\pi}(f(x+h)-f(x-h))^2 dx \; (h>0)$$

is uniformly bounded in h. [K. Tandori]

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9 Let $f(t)$ be a continuous function on the interval $0 \le t \le 1$, and define the two sets of points

$$A_t=\{(t,0): t\in [0,1]\}, B_t=\{(f(t),1): t\in[0,1]\}.$$

Show that the union of all segments $\overline{A_tB_t}$ is Lebesgue-measurable, and find the minimum of its measure with respect to all functions f. [A. Csaszar]

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10 Select n points on a circle independently with uniform distribution. Let P_n be the probability that the center of the
circle is in the interior of the convex hull of these n points. Calculate the probabilities P_3 and P_4. [A. Renyi]
