Mikls Schweitzer 1964

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by ehsan2004

1 Among all possible representations of the positive integer n as $n = \sum_{i=1}^{k} a_i$ with positive integers $k, a_1 < a_2 < ... < a_k$, when will the product $\prod_{i=1}^{k} a_i$ be maximum?

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2 Let p be a prime and let

$$l_k(x,y) = a_k x + b_k y \ (k = 1, 2, ..., p^2) \ .$$

be homogeneous linear polynomials with integral coefficients. Suppose that for every pair (ξ, η) of integers, not both divisible by p, the values $l_k(\xi, \eta), \ 1 \leq k \leq p^2$, represent every residue class $\bmod p$ exactly p times. Prove that the set of pairs $\{(a_k, b_k) : 1 \leq k \leq p^2\}$ is identical $\bmod p$ with the set $\{(m, n) : 0 \leq m, n \leq p-1\}$.

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3 Prove that the intersection of all maximal left ideals of a ring is a (two-sided) ideal.

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4 Let $A_1, A_2, ..., A_n$ be the vertices of a closed convex n-gon K numbered consecutively. Show that at least $n-3$
vertices A_i have the property that the reflection of A_i with respect to the midpoint of $A_{i-1}A_{i+1}$ is contained in K. (Indices are meant $\bmod n$.)

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5 Is it true that on any surface homeomorphic to an open disc there exist two congruent curves homeomorphic to a circle?

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6 Let $y_1(x)$ be an arbitrary, continuous, positive function on $[0, A]$, where A is an arbitrary positive number. Let

$$y_{n+1} = 2 \int_0^x \sqrt{y_n(t)} dt \ (n = 1, 2, ...) \ .$$

Prove that the functions $y_n(x)$ converge to the function $y = x^2$ uniformly on $[0, A]$.

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7 Find all linear homogeneous differential equations with continuous coefficients (on the whole real line) such that for any solution $f(t)$ and any real number c, $f(t+c)$ is also a solution.

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8 Let F be a closed set in the n-dimensional Euclidean space. Construct a function that is 0 on F, positive outside F , and whose partial derivatives all exist.

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9 Let E be the set of all real functions on $I = [0, 1]$. Prove that one cannot define a topology on E in which $f_n \to f$ holds if and only if f_n converges to f almost everywhere.

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10 Let $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2n}$ be independent random variables such that $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$ for all i, and define $S_k = \sum_{i=1}^{k} \varepsilon_i,\ 1 \leq k \leq 2n$. Let N_{2n} denote the number of integers $k \in [2, 2n]$ such that either $S_k > 0$, or $S_k = 0$ and $S_{k-1} > 0$. Compute the variance of N_{2n}.