Mikls Schweitzer 1965

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by ehsan2004

1 Let p be a prime, n a natural number, and S a set of cardinality p^n . Let **P** be a family of partitions of S into nonempty parts of sizes divisible by p such that the intersection of any two parts that occur in any of the partitions has at most one element. How large can $|$**P**$|$ be?

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2 Let R be a finite commutative ring. Prove that R has a multiplicative identity element (1) if and only if the annihilator of R is 0 (that is, $aR=0,\ a\in R$ imply $a=0$).

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3 Let $a, b_0, b_1, b_2, ..., b_{n-1}$ be complex numbers, A a complex square matrix of order p, and E the unit matrix of order p. Assuming that the eigenvalues of A are given, determine the eigenvalues of the matrix

$$B=\begin{pmatrix} b_0E & b_1A & b_2A^2 & \cdots & b_{n-1}A^{n-1} \\ ab_{n-1}A^{n-1} & b_0E & b_1A & \cdots & b_{n-2}A^{n-2} \\ ab_{n-2}A^{n-2} & ab_{n-1}A^{n-1} & b_0E & \cdots & b_{n-3}A^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ab_1A & ab_2A^2 & ab_3A^3 & \cdots & b_0E \end{pmatrix}$$

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4 The plane is divided into domains by n straight lines in general position, where $n \geq 3$. Determine the maximum and minimum possible number of angular domains among them. (We say that n lines are in general position if no two are parallel and no three are concurrent.)

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5 Let $A=A_1A_2A_3A_4$ be a tetrahedron, and suppose that for each $j\neq k$, $[A_j,A_{jk}]$ is a segment of length ρ extending from A_j in the direction of A_k. Let p_j be the intersection line of the planes $[A_{jk}A_{jl}A_{jm}]$ and $[A_kA_lA_m]$. Show that there are infinitely many straight lines that intersect the straight lines p_1,p_2,p_3,p_4 simultaneously.

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6 Consider the radii of normal curvature of a surface at one of its points P_0 in two conjugate direction (with respect to the Dupin indicatrix). Show that their sum does not depend on the choice of the conjugate directions. (We exclude the choice of asymptotic directions in the case of a hyperbolic point.)

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7 Prove that any uncountable subset of the Euclidean n-space contains an countable subset with the property that the distances between different pairs of points are different (that is, for any points $P_1\neq P_2$ and $Q_1\neq Q_2$ of this subset, $\overline{P_1P_2}=\overline{Q_1Q_2}$ implies either $P_1=Q_1$ and $P_2=Q_2$, or $P_1=Q_2$ and $P_2=Q_1$). Show that a similar statement is not valid if the Euclidean n-space is replaced with a (separable) Hilbert space.

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8 Let the continuous functions $f_n(x),\ n=1,2,3,...,$ be defined on the interval $[a,b]$ such that every point of $[a,b]$ is a root of $f_n(x)=f_m(x)$ for some $n\neq m$. Prove that there exists a subinterval of $[a,b]$ on which two of the functions are equal.

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9 Let f be a continuous, nonconstant, real function, and assume the existence of an F such that $f(x+y)=F[f(x),f(y)]$ for all real x and y. Prove that f is strictly monotone.

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10 A gambler plays the following coin-tossing game. He can bet an arbitrary positive amount of money. Then a fair coin is tossed, and the gambler wins or loses the amount he bet depending on the outcome. Our gambler, who starts playing with x forints, where $0<x<2C$, uses the following strategy: if at a given time his capital is $y<C$, he risks all of it; and if he has $y>C$, he only bets $2C-y$. If he has exactly $2C$ forints, he stops playing. Let $f(x)$ be the probability that he reaches $2C$ (before going bankrupt). Determine the value of $f(x)$.
