Mikls Schweitzer 1967

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by ehsan2004

1 Let
$$f(x)=a_0+a_1x+a_2x^2+a_{10}x^{10}+a_{11}x^{11}+a_{12}x^{12}+a_{13}x^{13}\ (a_{13}\neq 0)$$
and
$$g(x)=b_0+b_1x+b_2x^2+b_3x^3+b_{11}x^{11}+b_{12}x^{12}+b_{13}x^{13}\ (b_3\neq 0)$$
be polynomials over the same field. Prove that the degree of their greatest common divisor is at least 6.

L. Redei

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2 Let K be a subset of a group G that is not a union of lift cosets of a proper subgroup. Prove that if G is a torsion group or if K is a finite set, then the subset
$$\bigcap_{k\in K} k^{-1}K$$
consists of the identity alone.

L. Redei

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3 Prove that if an infinite, noncommutative group G contains a proper normal subgroup with a commutative factor group, then G also contains an infinite proper normal subgroup.

B. Csakany

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4 Let $a_1,a_2,...,a_N$ be positive real numbers whose sum equals 1. For a natural number i, let n_i denote the number
of a_k for which $2^{1-i}\geq a_k\geq 2^{-i}$ holds. Prove that
$$\sum_{i=1}^{\infty}\sqrt{n_i2^{-i}}\leq 4+\sqrt{\log_2 N}.$$

L. Leinder

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5 Let f be a continuous function on the unit interval $[0,1]$. Show that
$$\lim_{n\to\infty}\int_0^1 ... \int_0^1 f(\frac{x_1+...+x_n}{n})dx_1...dx_n=f(\frac{1}{2})$$

and

$$\lim_{n\to\infty} \int_0^1 ... \int_0^1 f(\sqrt[n]{x_1...x_n})dx_1...dx_n = f(\frac{1}{e}).$$

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6 Let A be a family of proper closed subspaces of the Hilbert space $H = l^2$ totally ordered with respect to inclusion (that is
, if $L_1, L_2 \in A$, then either $L_1 \subset L_2$ or $L_2 \subset L_1$). Prove that there exists a vector $x \in H$ not contaied in any of the subspaces L belonging to A.

B. Szokefalvi Nagy

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7 Let U be an $n \times n$ orthogonal matrix. Prove that for any $n \times n$ matrix A, the matrices

$$A_m = \frac{1}{m+1} \sum_{j=0}^{m} U^{-j} A U^j$$

converge entrywise as $m \to \infty$.

L. Kovacs

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8 Suppose that a bounded subset S of the plane is a union of congruent, homothetic, closed triangles. Show that the boundary of S can be covered by a finite number of rectifiable arcs.

L. Geher

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9 Let F be a surface of nonzero curvature that can be represented around one of its points P by a power series and is symmetric around the normal planes parallel to the principal directions at P. Show that the derivative with respect to the arc length of the curvature of an arbitrary normal section at P vanishes at P. Is it possible to replace the above symmetry condition by a weaker one?

A. Moor

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10 Let $\sigma(S_n, k)$ denote the sum of the kth powers of the lengths of the sides of the convex n-gon S_n inscribed in a unit circle. Show that for any natural number greater than 2 there exists a real number k_0 between 1 and 2 such that $\sigma(S_n, k_0)$ attains its maximum for the regular n-gon.

L. Fejes Toth
