Mikls Schweitzer 1968

www.artofproblemsolving.com/community/c3702

by ehsan2004

1 Consider the endomorphism ring of an Abelian torsion-free (resp. torsion) group G. Prove that this ring is Neumann-regular if and only if G is a discrete direct sum of groups isomorphic to the additive group of the rationals (resp. ,a discrete direct sum of cyclic groups of prime order). (A ring R is called Neumann-regular if for every $\alpha \in R$ there exists a $\beta \in R$ such that $\alpha\beta\alpha = \alpha$.)

E. Freid

---

2 Let $a_1, a_2, ..., a_n$ be nonnegative real numbers. Prove that

$$(\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n} a_i^{n-1}) \leq n \prod_{i=1}^{n} a_i + (n-1)(\sum_{i=1}^{n} a_i^n).$$

J. Suranyi

---

3 Let K be a compact topological group, and let F be a set of continuous functions defined on K that has cardinality greater that continuum. Prove that there exist $x_0 \in K$ and $f \neq g \in F$ such that

$$f(x_0) = g(x_0) = \max_{x \in K} f(x) = \max_{x \in K} g(x).$$

I. Juhasz

---

4 Let f be a complex-valued, completely multiplicative,arithmetical function. Assume that there exists an infinite increasing sequence N_k of natural numbers such that

$$f(n) = A_k \neq 0 \text{ provided } N_k \leq n \leq N_k + 4\sqrt{N_k}\ .$$

Prove that f is identically 1.

I. Katai

---

5 Let k be a positive integer, z a complex number, and $\varepsilon < \frac{1}{2}$ a positive number. Prove that the following inequality holds for infinitely many positive integers n:

$$| \sum_{0 \leq l \leq \frac{n}{k+1}} \binom{n-kl}{l} z^l | \geq (\frac{1}{2} - \varepsilon)^n.$$

P. Turan

---

6 Let $\Psi = \langle A; ...\rangle$ be an arbitrary, countable algebraic structure (that is, Ψ can have an arbitrary number of finitary operations and relations). Prove that Ψ has as many as continuum automorphisms if and only if for any finite subset A' of A there is an automorphism $\pi_{A'}$ of Ψ different from the identity automorphism and such that

$$(x)\pi_{A'} = x$$

for every $x \in A'$.

M. Makkai

---

7 For every natural number r, the set of r-tuples of natural numbers is partitioned into finitely many classes. Show that if $f(r)$ is a function such that $f(r) \geq 1$ and $\lim_{r\to\infty} f(r) = +\infty$, then there exists an infinite set of natural numbers that, for all r, contains r-triples from at most $f(r)$ classes. Show that if $f(r) \nrightarrow +\infty$, then there is a family of partitions such that no such infinite set exists.

P. Erdos, A. Hajnal

---

8 Let n and k be given natural numbers, and let A be a set such that

$$|A| \leq \frac{n(n+1)}{k+1}.$$

For $i = 1, 2, ..., n+1$, let A_i be sets of size n such that

$$|A_i \cap A_j| \leq k \; (i \neq j) \; ,$$

$$A = \bigcup_{i=1}^{n+1} A_i.$$

Determine the cardinality of A.

K. Corradi

---

9 Let $f(x)$ be a real function such that

$$\lim_{x\to+\infty} \frac{f(x)}{e^x} = 1$$

and $|f''(x)| \leq c|f'(x)|$ for all sufficiently large x. Prove that

$$\lim_{x\to+\infty} \frac{f'(x)}{e^x} = 1.$$

P. Erdos

---

10 Let h be a triangle of perimeter 1, and let H be a triangle of perimeter λ homothetic to h. Let $h_1, h_2, ...$ be translates of h such that , for all i, h_i is different from h_{i+2} and touches H and h_{i+1} (that is, intersects without overlapping). For which values of λ can these triangles be chosen so that the sequence $h_1, h_2, ...$ is periodic? If $\lambda \geq 1$ is such a value, then determine the number of different triangles in a periodic
chain $h_1, h_2, ...$ and also the number of times such a chain goes around the triangle H.

L. Fejes-Toth

---

11 Let $A_1, ..., A_n$ be arbitrary events in a probability field. Denote by C_k the event that at least k of $A_1, ..., A_n$ occur. Prove that

$$\prod_{k=1}^{n} P(C_k) \leq \prod_{k=1}^{n} P(A_k).$$

A. Renyi
