Mikls Schweitzer 1969

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by ehsan2004

1 Let G be an infinite group generated by nilpotent normal subgroups. Prove that every maximal Abelian normal subgroup of G is infinite. (We call an Abelian normal subgroup maximal if it is not contained in another Abelian normal subgroup.)

P. Erdos

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2 Let $p \geq 7$ be a prime number, ζ a primitive pth root of unity, c a rational number. Prove that in the additive group generated by the numbers $1, \zeta, \zeta^2, \zeta^3 + \zeta^{-3}$ there are only finitely many elements whose norm is equal to c. (The norm is in the pth cyclotomic field.)

K. Gyory

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3 Let $f(x)$ be a nonzero, bounded, real function on an Abelian group G, $g_1, ..., g_k$ are given elements of G and $\lambda_1, ..., \lambda_k$ are real numbers. Prove that if

$$\sum_{i=1}^{k} \lambda_i f(g_i x) \geq 0$$

holds for all $x \in G$, then

$$\sum_{i=1}^{k} \lambda_i \geq 0.$$

A. Mate

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4 Show that the following inequality hold for all $k \geq 1$, real numbers $a_1, a_2, ..., a_k$, and positive numbers $x_1, x_2, ..., x_k$.

$$\ln \frac{\sum\limits_{i=1}^{k} x_i}{\sum\limits_{i=1}^{k} x_i^{1-a_i}} \leq \frac{\sum\limits_{i=1}^{k} a_i x_i \ln x_i}{\sum\limits_{i=1}^{k} x_i}.$$

L. Losonczi

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5 Find all continuous real functions f, g and h defined on the set of positive real numbers and satisfying the relation

$$f(x+y) + g(xy) = h(x) + h(y)$$

for all $x > 0$ and $y > 0$.

Z. Daroczy

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6 Let x_0 be a fixed real number, and let f be a regular complex function in the half-plane $\Re z > x_0$ for which there exists a nonnegative function $F \in L_1(-\infty, +\infty)$ satisfying $|f(\alpha + i\beta)| \leq F(\beta)$ whenever $\alpha > x_0$, $-\infty < \beta < +\infty$. Prove that

$$\int_{\alpha - i\infty}^{\alpha + i\infty} f(z)dz = 0.$$

L. Czach

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7 Prove that if a sequence of Mikusinski operators of the form $\mu e^{-\lambda s}$ (λ and μ nonnegative real numbers, s the differentiation operator) is convergent in the sense of Mikusinski, then its limit is also of this form.

E. Geaztelyi

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8 Let f and g be continuous positive functions defined on the interval $[0, +\infty)$, and let $E \subset [0, +\infty)$ be a set of positive measure. Prove that the range of the function defined on $E \times E$ by the relation

$$F(x, y) = \int_0^x f(t)dt + \int_0^y g(t)dt$$

has a nonvoid interior.

L. Losonczi

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9 In n-dimensional Euclidean space, the union of any set of closed balls (of positive radii) is measurable in the sense of Lebesgue.

A. Csaszar

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10 In n-dimensional Euclidean space, the square of the two-dimensional Lebesgue measure of a bounded, closed, (two-dimensional) planar set is equal to the sum of the squares of the measures of the orthogonal projections of the given set on the n-coordinate hyperplanes.

L. Tamassy

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11 Let $A_1, A_2, ...$ be a sequence of infinite sets such that $|A_i \cap A_j| \leq 2$ for $i \neq j$. Show that the sequence of indices can be divided into two disjoint sequences $i_1 < i_2 < ...$ and $j_1 < j_2 < ...$ in such a way that, for some sets E and F, $|A_{i_n} \cap E| = 1$ and $|A_{j_n} \cap F| = 1$ for $n = 1, 2, ...$

P. Erdos

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12 Let A and B be nonsingular matrices of order p, and let ξ and η be independent random vectors of dimension p. Show that if ξ, η and $\xi A + \eta B$ have the same distribution, if their first and second moments exist, and if their covariance matrix is the identity matrix, then these random vectors are normally distributed.

B. Gyires
