Mikls Schweitzer 1970

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by ehsan2004

1 We have $2n+1$ elements in the commutative ring R:

$$\alpha, \alpha_1, ..., \alpha_n, \varrho_1, ..., \varrho_n.$$

Let us define the elements

$$\sigma_k = k\alpha + \sum_{i=1}^{n} \alpha_i \varrho_i^k.$$

Prove that the ideal $(\sigma_0, \sigma_1, ..., \sigma_k, ...)$ can be finitely generated.

L. Redei

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2 Let G and H be countable Abelian p-groups (p an arbitrary prime). Suppose that for every positive integer n,

$$p^n G \neq p^{n+1} G.$$

Prove that H is a homomorphic image of G.

M. Makkai

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3 The traffic rules in a regular triangle allow one to move only along segments parallel to one of the altitudes of the triangle. We define the distance between two points of the triangle to be the length of the shortest such path between them. Put $\binom{n+1}{2}$ points into the triangle in such a way that the minimum distance between pairs of points is maximal.

L. Fejes-Toth

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4 If c is a positive integer and p is an odd prime, what is the smallest residue (in absolute value) of

$$\sum_{n=0}^{\frac{p-1}{2}} \binom{2n}{n} c^n \pmod{p}\ ?$$

J. Suranyi

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5 Prove that two points in a compact metric space can be joined with a rectifiable arc if and only if there exists a positive number K such that, for any $\varepsilon > 0$, these points can be connected with an ε-chain not longer that K.

M. Bognar

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6 Let a neighborhood basis of a point x of the real line consist of all Lebesgue-measurable sets containing x whose density at x equals 1. Show that this requirement defines a topology that is regular but not normal.

A. Csaszar

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7 Let us use the word N-measure for nonnegative, finitely additive set functions defined on all subsets of the positive integers, equal to 0 on finite sets, and equal to 1 on the whole set. We say that the system Υ of sets determines the N-measure μ if any N-measure coinciding with μ on all elements of Υ is necessarily identical with μ.
Prove the existence of an N-measure μ that cannot be determined by a system of cardinality less than continuum.

I. Juhasz

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8 Let $\pi_n(x)$ be a polynomial of degree not exceeding n with real coefficients such that

$$|\pi_n(x)| \le \sqrt{1-x^2} \text{ for } \ -1 \le x \le 1 \ .$$

Then

$$|\pi_n'(x)| \le 2(n-1).$$

P. Turan

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9 Construct a continuous function $f(x)$, periodic with period 2π, such that the Fourier series of $f(x)$ is divergent at $x=0$, but the Fourier series of $f^2(x)$ is uniformly convergent on $[0,2\pi]$.

P. Turan

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10 Prove that for every ϑ, $0<\vartheta<1$, there exist a sequence λ_n of positive integers and a series $\sum_{n=1}^{\infty} a_n$ such that

(i)$\lambda_{n+1}-\lambda_n > (\lambda_n)^{\vartheta}$,

(ii) $\lim_{r\to 1^-} \sum_{n=1}^{\infty} a_n r^{\lambda_n}$ exists,

(iii) $\sum_{n=1}^{\infty} a_n$ is divergent.

P. Turan

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11 Let $\xi_1, \xi_2, ...$ be independent random variables such that $E\xi_n = m > 0$ and $\mathrm{Var}(\xi_n) = \sigma^2 < \infty$ $(n = 1, 2, ...)$. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Prove that

$$P\left(\lim_{n\to\infty} \sum_{k=1}^{n} a_k \xi_k = \infty\right) = 1.$$

P. Revesz

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12 Let $\vartheta_1, ..., \vartheta_n$ be independent, uniformly distributed, random variables in the unit interval $[0, 1]$. Define

$$h(x) = \frac{1}{n} \#\{k :\ \vartheta_k < x\ \}.$$

Prove that the probability that there is an $x_0 \in (0, 1)$ such that $h(x_0) = x_0$, is equal to $1 - \frac{1}{n}$.

G. Tusnady
