Mikls Schweitzer 1971
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by ehsan2004

1 Let G be an infinite compact topological group with a Hausdorff topology. Prove that G contains an element $g \neq 1$ such that the set of all powers of g is either everywhere dense in G or nowhere dense in G.

J. Erdos

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2 Prove that there exists an ordered set in which every uncountable subset contains an uncountable, well-ordered subset and that cannot be represented as a union of a countable family of well-ordered subsets.

A. Hajnal

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3 Let $0 < a_k < 1$ for $k = 1, 2, \ldots$. Give a necessary and sufficient condition for the existence, for every $0 < x < 1$, of a permutation π_x of the positive integers such that

$$x = \sum_{k=1}^{\infty} \frac{a_{\pi_x}(k)}{2^k}.$$

P. Erdos

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4 Suppose that V is a locally compact topological space that admits no countable covering with compact sets. Let **C**
denote the set of all compact subsets of the space V and **U** the set of open subsets that are not contained in any compact set. Let f be a function from **U** to **C** such that $f(U) \subseteq U$ for all $U \in$ **U**. Prove that either

(i) there exists a nonempty compact set C such that $f(U)$ is not a proper subset of C whenever $C \subseteq U \in$ **U**,

(ii) or for some compact set C, the set

$$f^{-1}(C) = \bigcup \{U \in \mathbf{U} \ : \ f(U) \subseteq C\}$$

is an element of **U**, that is, $f^{-1}(C)$ is not contained in any compact set.

A. Mate

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5 Let $\lambda_1 \le \lambda_2 \le ...$ be a positive sequence and let K be a constant such that

$$\sum_{k=1}^{n-1} \lambda_k^2 < K\lambda_n^2 \ (n = 1, 2, ...).$$

Prove that there exists a constant K' such that

$$\sum_{k=1}^{n-1} \lambda_k < K'\lambda_n \ (n = 1, 2, ...).$$

L. Leindler

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6 Let $a(x)$ and $r(x)$ be positive continuous functions defined on the interval $[0, \infty)$, and let

$$\liminf_{x\to\infty}(x - r(x)) > 0.$$

Assume that $y(x)$ is a continuous function on the whole real line, that it is differentiable on $[0, \infty)$, and that it satisfies

$$y'(x) = a(x)y(x - r(x))$$

on $[0, \infty)$. Prove that the limit

$$\lim_{x\to\infty} y(x) \exp\left\{-\int_0^x a(u)du\right\}$$

exists and is finite.

I. Gyori

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7 Let $n \ge 2$ be an integer, let S be a set of n elements, and let $A_i, \ 1 \le i \le m$, be distinct subsets of S of size at least 2 such that

$$A_i \cap A_j \neq \emptyset, A_i \cap A_k \neq \emptyset, A_j \cap A_k \neq \emptyset, \text{ imply } \ A_i \cap A_j \cap A_k \neq \emptyset \ .$$

Show that $m \le 2^{n-1} - 1$.

P. Erdos

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8 Show that the edges of a strongly connected bipolar graph can be oriented in such a way that for any edge e there is a simple directed path from pole p to pole q containing e. (A strongly connected bipolar graph is a finite connected graph with two special vertices p and q having the property that there are no points $x, y, x \neq y$, such that all paths from x to p as well as all paths from x to q contain y.)

A. Adam

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9 Given a positive, monotone function $F(x)$ on $(0, \infty)$ such that $F(x)/x$ is monotone nondecreasing and $F(x)/x^{1+d}$ is monotone nonincreasing for some positive d, let $\lambda_n > 0$ and $a_n \geq 0,\ n \geq 1$. Prove that if

$$\sum_{n=1}^{\infty} \lambda_n F\left(a_n \sum_{k=1}^{n} \frac{\lambda_k}{\lambda_n}\right) < \infty,$$

or

$$\sum_{n=1}^{\infty} \lambda_n F\left(\sum_{k=1}^{n} a_k \frac{\lambda_k}{\lambda_n}\right) < \infty,$$

then $\sum_{n=1}^{\infty} a_n$ is convergent.

L. Leindler

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10 Let $\{\phi_n(x)\}$ be a sequence of functions belonging to $L^2(0,1)$ and having norm less that 1 such that for any
subsequence $\{\phi_{n_k}(x)\}$ the measure of the set

$$\{x \in (0,1) :\ |\frac{1}{\sqrt{N}} \sum_{k=1}^{N} \phi_{n_k}(x)| \geq y\ \}$$

tends to 0 as y and N tend to infinity. Prove that ϕ_n tends to 0 weakly in the function space $L^2(0,1)$.

F. Moricz

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11 Let C be a simple arc with monotone curvature such that C is congruent to its evolute. Show that under appropriate differentiability conditions, C is a part of a cycloid or a logarithmic spiral with polar equation $r = ae^{\vartheta}$.

J. Szenthe
