Mikls Schweitzer 1972

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by ehsan2004

1 Let $\mathcal{F}$ be a nonempty family of sets with the following properties:

(a) If $X \in \mathcal{F}$, then there are some $Y \in \mathcal{F}$ and $Z \in \mathcal{F}$ such that $Y \cap Z = \emptyset$ and $Y \cup Z = X$.

(b) If $X \in \mathcal{F}$, and $Y \cup Z = X, Y \cap Z = \emptyset$, then either $Y \in \mathcal{F}$ or $Z \in \mathcal{F}$.

Show that there is a decreasing sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq$ ... of sets $X_n \in \mathcal{F}$ such that

$$\bigcap_{n=0}^{\infty} X_n = \emptyset.$$

F. Galvin

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2 Let $\leq$ be a reflexive, antisymmetric relation on a finite set A. Show that this relation can be extended to an appropriate finite superset B of A such that $\leq$ on B remains reflexive, antisymmetric, and any two elements of B have a least upper bound as well as a greatest lower bound. (The relation $\leq$ is extended to B if for $x, y \in A, x \leq y$ holds in A if and only if it holds in B.)

E. Freid

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3 Let $\lambda_i \ (i=1,2,...)$ be a sequence of distinct positive numbers tending to infinity. Consider the set of all numbers representable in the form

$$\mu = \sum_{i=1}^{\infty} n_i \lambda_i,$$

where $n_i \geq 0$ are integers and all but finitely many n_i are 0. Let

$$L(x) = \sum_{\lambda_i \leq x} 1 \text{ and } M(x) = \sum_{\mu \leq x} 1 \, .$$

(In the latter sum, each μ occurs as many times as its number of representations in the above form.) Prove that if

$$\lim_{x \to \infty} \frac{L(x+1)}{L(x)} = 1,$$

then

$$\lim_{x \to \infty} \frac{M(x+1)}{M(x)} = 1.$$

G. Halasz

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4 Let G be a solvable torsion group in which every Abelian subgroup is finitely generated. Prove that G is finite.

J. Pelikan

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5 We say that the real-valued function $f(x)$ defined on the interval $(0,1)$ is approximately continuous on $(0,1)$ if for any $x_0 \in (0,1)$ and $\varepsilon > 0$ the point x_0 is a point of interior density 1 of the set

$$H = \{x: \ |f(x) - f(x_0)| < \varepsilon \}.$$

Let $F \subset (0,1)$ be a countable closed set, and $g(x)$ a real-valued function defined on F. Prove the existence of an approximately continuous function $f(x)$ defined on $(0,1)$ such that

$$f(x) = g(x) \text{ for all } \ x \in F \ .$$

M. Laczkovich, Gy. Petruska

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6 Let $P(z)$ be a polynomial of degree n with complex coefficients,

$$P(0) = 1, \text{ and } \ |P(z)| \leq M \ \text{ for } \ |z| \leq 1 \ .$$

Prove that every root of $P(z)$ in the closed unit disc has multiplicity at most $c\sqrt{n}$, where $c = c(M) > 0$ is a constant depending only on M.

G. Halasz

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7 Let $f(x,y,z)$ be a nonnegative harmonic function in the unit ball of $\mathbb{R}^3$ for which the inequality $f(x_0,0,0) \leq \varepsilon^2$ holds for some $0 \leq x_0 \leq 1$ and $0 < \varepsilon < (1-x_0)^2$. Prove that $f(x,y,z) \leq \varepsilon$ in the ball with center at the origin an radius $(1 - 3\varepsilon^{1/4})$.

P. Turan

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8 Given four points A_1, A_2, A_3, A_4 in the plane in such a way that A_4 is the centroid of the $\triangle A_1A_2A_3$, find a point A_5 in the plane that maximizes the ratio

$$\frac{\min_{1 \leq i < j < k \leq 5} T(A_iA_jA_k)}{\max_{1 \leq i < j < k \leq 5} T(A_iA_jA_k)}.$$

($T(ABC)$ denotes the area of the triangle $\triangle ABC$.)

J. Suranyi

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9 Let K be a compact convex body in the n-dimensional Euclidean space. Let $P_1, P_2, ..., P_{n+1}$ be the vertices of a simplex having maximal volume among all simplices inscribed in K. Define the points $P_{n+2}, P_{n+3}, ...$ successively so that $P_k \ (k > n+1)$ is a point of K for which the volume of the convex hull of $P_1, ..., P_k$ is maximal. Denote this volume by V_k. Decide, for different values of n, about the truth of the statement "the sequence $V_{n+1}, V_{n+2}, ...$ is concave."

L. Fejes- Toth, E. Makai

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10 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be second-countable topologies on the set E. We would like to find a real function σ defined on $E \times E$ such that

$$0 \leq \sigma(x,y) < +\infty, \ \sigma(x,x) = 0 \ ,$$

$$\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z) \ (x,y,z \in E) \ ,$$

and, for any $p \in E$, the sets

$$V_1(p,\varepsilon) = \{x : \ \sigma(x,p) < \varepsilon \ \} \ (\varepsilon > 0)$$

form a neighborhood base of p with respect to $\mathcal{T}_1$, and the sets

$$V_2(p,\varepsilon) = \{x : \ \sigma(p,x) < \varepsilon \ \} \ (\varepsilon > 0)$$

form a neighborhood base of p with respect to $\mathcal{T}_2$. Prove that such a function σ exists if and only if, for any $p \in E$ and $\mathcal{T}_i$-open set $G \ni p \ (i = 1, 2)$, there exist a $\mathcal{T}_i$-open set G' and a $\mathcal{T}_{3-i}$-closed set F with $p \in G' \subset F \subset G$.

A. Csaszar

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11 We throw N balls into n urns, one by one, independently and uniformly. Let $X_i = X_i(N,n)$ be the total number of balls in
the ith urn. Consider the random variable

$$y(N,n) = \min_{1 \leq i \leq n} |X_i - \frac{N}{n}|.$$

Verify the following three statements:

(a) If $n \to \infty$ and $N/n^3 \to \infty$, then

$$P\left(\frac{y(N,n)}{\frac{1}{n}\sqrt{\frac{N}{n}}} < x\right) \to 1 - e^{-x\sqrt{2/\pi}} \text{ for all } \ x > 0 \ .$$

(b) If $n \to \infty$ and $N/n^3 \leq K$ (K constant), then for any $\varepsilon > 0$ there is an $A > 0$ such that

$$P(y(N,n) < A) > 1 - \varepsilon.$$

(c) If $n \to \infty$ and $N/n^3 \to 0$ then

$$P(y(N,n) < 1) \to 1.$$

P. Revesz