**Mikls Schweitzer 1974**

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by ehsan2004

**1** Let $\mathcal{F}$ be a family of subsets of a ground set $X$ such that $\cup_{F \in \mathcal{F}} F = X$, and

(a) if $A, B \in \mathcal{F}$, then $A \cup B \subseteq C$ for some $C \in \mathcal{F}$;

(b) if $A_n \in \mathcal{F} \ (n = 0, 1, ...)$ , $B \in \mathcal{F}$, and $A_0 \subset A_1 \subset$ ..., then, for some $k \geq 0, \ A_n \cap B = A_k \cap B$ for all $n \geq k$.

Show that there exist pairwise disjoint sets $X_\gamma$ $(\gamma \in \Gamma \ )$, with $X = \cup \{X_\gamma : \ \gamma \in \Gamma \ \}$, such that every $X_\gamma$ is contained in some member of $\mathcal{F}$, and every element of $\mathcal{F}$ is contained in the union of finitely many $X_\gamma$'s.

*A. Hajnal*

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**2** Let $G$ be a $2$-connected nonbipartite graph on $2n$ vertices. Show that the vertex set of $G$ can be split into two classes of $n$ elements such that the edges joining the two classes form a connected, spanning subgraph.

*L. Lovasz*

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**3** Prove that a necessary and sufficient for the existence of a set $S \subset \{1, 2, ..., n\}$ with the property that the integers $0, 1, ..., n-1$ all have an odd number of representations in the form $x - y, x, y \in S$, is that $(2n - 1)$ has a multiple of the form $2.4^k - 1$

*L. Lovasz, J. Pelikan*

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**4** Let $R$ be an infinite ring such that every subring of $R$ different from $\{0\}$ has a finite index in $R$. (By the index of a subring, we mean the index of its additive group in the additive group of $R$.) Prove that the additive group of $R$ is cyclic.

*L. Lovasz, J. Pelikan*

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**5** Let $\{f_n\}_{n=0}^{\infty}$ be a uniformly bounded sequence of real-valued measurable functions defined on $[0, 1]$ satisfying

$$\int_0^1 f_n^2 = 1.$$

Further, let $\{c_n\}$ be a sequence of real numbers with

$$\sum_{n=0}^{\infty} c_n^2 = +\infty.$$

Prove that some re-arrangement of the series $\sum_{n=0}^{\infty} c_n f_n$ is divergent on a set of positive measure.

*J. Komlos*

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**6** Let $f(x) = \sum_{n=1}^{\infty} a_n/(x+n^2), \ \ (x \geq 0)$ , where $\sum_{n=1}^{\infty} |a_n| n^{-\alpha} < \infty$ for some $\alpha > 2$. Let us assume that for some $\beta > 1/\alpha$, we have $f(x) = O(e^{-x^{\beta}})$ as $x \to \infty$. Prove that $a_n$ is identically $0$.

*G. Halasz*

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**7** Given a positive integer $m$ and $0 < \delta < \pi$, construct a trigonometric polynomial $f(x) = a_0 + \sum_{n=1}^{m} (a_n \cos nx + b_n \sin nx)$ of degree $m$ such that $f(0) = 1, \int_{\delta \leq |x| \leq \pi} |f(x)| dx \leq c/m$, and $\max_{-\pi \leq x \leq \pi} |f'(x)| \leq c/\delta$, for some universal constant $c$.

*G. Halasz*

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**8** Prove that there exists a topological space $T$ containing the real line as a subset, such that the Lebesgue-measurable functions, and only those, extend continuously over $T$. Show that the real line cannot be an everywhere-dense subset of such a space $T$.

*A. Csaszar*

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**9** Let $A$ be a closed and bounded set in the plane, and let $C$ denote the set of points at a unit distance from $A$. Let $p \in C$, and assume that the intersection of $A$ with the unit circle $K$ centered at $p$ can be covered by an arc shorter that a semicircle of $K$. Prove that the intersection of $C$ with a suitable neighborhood of $p$ is a simple arc which $p$ is not an endpoint.

*M. Bognar*

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**10** Let $\mu$ and $\nu$ be two probability measures on the Borel sets of the plane. Prove that there are random variables $\xi_1, \xi_2, \eta_1, \eta_2$ such that

(a) the distribution of $(\xi_1, \xi_2)$ is $\mu$ and the distribution of $(\eta_1, \eta_2)$ is $\nu$,

(b) $\xi_1 \leq \eta_1, \xi_2 \leq \eta_2$ almost everywhere, if an only if $\mu(G) \geq \nu(G)$ for all sets of the form $G = \cup_{i=1}^{k} (-\infty, x_i) \times (-\infty, y_i)$.

*P. Major*

 