Mikls Schweitzer 1975

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by ehsan2004

1 Show that there exists a tournament $(T, \to)$ of cardinality $\aleph_1$ containing no transitive subtournament of size $\aleph_1$. (A structure $(T, \to)$ is a *tournament* if $\to$ is a binary, irreflexive, asymmetric and trichotomic relation. The tournament $(T, \to)$ is transitive if $\to$ is transitive, that is, if it orders T.)

A. Hajnal

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2 Let $\mathcal{A}_n$ denote the set of all mappings $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that $f^{-1}(i) := \{k\colon f(k) = i\ \} \neq \varnothing$ implies $f^{-1}(j) \neq \varnothing, j \in \{1, 2, \dots, i\}$. Prove

$$|\mathcal{A}_n| = \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}}.$$

L. Lovasz

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3 Let S be a semigroup without proper two-sided ideals and suppose that for every $a, b \in S$ at least one of the products ab and ba is equal to one of the elements a, b. Prove that either $ab = a$ for all $a, b \in S$ or $ab = b$ for all $a, b \in S$.

L. Megyesi

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4 Prove that the set of rational-valued, multiplicative arithmetical functions and the set of complex rational-valued, multiplicative arithmetical functions form isomorphic groups with the convolution operation $f \circ g$ defined by

$$(f \circ g)(n) = \sum_{d|n} f(d) g(\frac{n}{d}).$$

(We call a complex number *complex rational*, if its real and imaginary parts are both rational.)

B. Csakany

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5 Let $\{f_n\}$ be a sequence of Lebesgue-integrable functions on $[0, 1]$ such that for any Lebesgue-measurable subset E of $[0, 1]$ the sequence $\int_E f_n$ is convergent. Assume also that $\lim_n f_n = f$ exists almost everywhere. Prove that f is integrable and $\int_E f = \lim_n \int_E f_n$. Is the assertion also true if E runs only over intervals but we also assume $f_n \geq 0$? What happens if $[0, 1]$ is replaced by $[0, +\infty)$?

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J. Szucs

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6 Let f be a differentiable real function and let M be a positive real number. Prove that if

$$|f(x+t)-2f(x)+f(x-t)| \leq Mt^2 \text{ for all } x \text{ and } t\ ,$$

then

$$|f'(x+t)-f'(x)| \leq M|t|.$$

J. Szabados

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7 Let $a<a'<b<b'$ be real numbers and let the real function f be continuous on the interval $[a,b']$ and differentiable in its interior. Prove that there exist $c \in (a,b), c' \in (a',b')$ such that

$$f(b)-f(a)=f'(c)(b-a),$$
$$f(b')-f(a')=f'(c')(b'-a'),$$

and $c<c'$.

B. Szokefalvi Nagy

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8 Prove that if

$$\sum_{n=1}^m a_n \leq Na_m \ (m=1,2,...)$$

holds for a sequence $\{a_n\}$ of nonnegative real numbers with some positive integer N, then $\alpha_{i+p} \geq p\alpha_i$ for $i,p=1,2,...$, where

$$\alpha_i=\sum_{n=(i-1)N+1}^{iN} a_n \ (i=1,2,...)\ .$$

L. Leindler

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9 Let l_0,c,α,g be positive constants, and let $x(t)$ be the solution of the differential equation

$$([l_0+ct^{\alpha}]^2x')'+g[l_0+ct^{\alpha}]\sin x=0, \ t \geq 0, \ \ -\frac{\pi}{2}<x<\frac{\pi}{2},$$

satisfying the initial conditions $x(t_0)=x_0, \ x'(t_0)=0$. (This is the equation of the mathematical pendulum whose length changes according to the law $l=l_0+ct^{\alpha}$.) Prove that $x(t)$ is defined on the interval $[t_0,\infty)$; furthermore, if $\alpha>2$ then for every $x_0 \neq 0$ there exists a t_0 such that

$$\liminf_{t \rightarrow \infty} |x(t)|>0.$$

L. Hatvani

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10 Prove that an idempotent linear operator of a Hilbert space is self-adjoint if and only if it has norm 0 or 1.

J. Szucs

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11 Let $X_1, X_2, ..., X_n$ be (not necessary independent) discrete random variables. Prove that there exist at least $n^2/2$ pairs (i,j) such that

$$H(X_i + X_j) \geq \frac{1}{3} \min_{1 \leq k \leq n} \{H(X_k)\},$$

where $H(X)$ denotes the Shannon entropy of X.

GY. Katona

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12 Assume that a face of a convex polyhedron P has a common edge with every other face. Show that there exists a simple closed polygon that consists of edges of P and passes through all vertices.

L .Lovasz

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