Mikls Schweitzer 1977

www.artofproblemsolving.com/community/c3711

by ehsan2004

1 Consider the intersection of an ellipsoid with a plane σ passing through its center O. On the line through the point O perpendicular to σ, mark the two points at a distance from O equal to the area of the intersection. Determine the loci of the marked points as σ runs through all such planes.

L. Tamassy

---

2 Construct on the real projective plane a continuous curve, consisting of simple points, which is not a straight line and is intersected in a single point by every tangent and every secant of a given conic.

F. Karteszi

---

3 Prove that if a, x, y are p-adic integers different from 0 and $p|x, pa|xy$, then

$$\frac{1}{y}\frac{(1+x)^y-1}{x} \equiv \frac{\log(1+x)}{x} \pmod{a}.$$

L. Redei

---

4 Let $p > 5$ be a prime number. Prove that every algebraic integer of the pth cyclotomic field can be represented as a sum of (finitely many) distinct units of the ring of algebraic integers of the field.

K. Gyory

---

5 Suppose that the automorphism group of the finite undirected graph $X = (P, E)$ is isomorphic to the quaternion group (of order 8). Prove that the adjacency matrix of X has an eigenvalue of multiplicity at least 4.

($P = \{1, 2, \dots, n\}$ is the set of vertices of the graph X. The set of edges E is a subset of the set of all unordered pairs of elements of P. The group of automorphisms of X consists of those permutations of P that map edges to edges. The adjacency matrix $M = [m_{ij}]$ is the $n \times n$ matrix defined by $m_{ij} = 1$ if $\{i, j\} \in E$ and $m_{i,j} = 0$ otherwise.)

L. Babai

---

6 Let f be a real function defined on the positive half-axis for which $f(xy) = xf(y) + yf(x)$ and $f(x+1) \leq f(x)$ hold for every positive x and y. Show that if $f(1/2) = 1/2$, then

$$f(x) + f(1-x) \geq -x \log_2 x - (1-x) \log_2 (1-x)$$

for every $x \in (0,1)$.

Z. Daroczy, Gy. Maksa

---

7 Let G be a locally compact solvable group, let $c_1, \dots, c_n$ be complex numbers, and assume that the complex-valued functions f and g on G satisfy

$$\sum_{k=1}^{n} c_k f(xy^k) = f(x)g(y) \text{ for all } x, y \in G \ .$$

Prove that if f is a bounded function and

$$\inf_{x \in G} \mathrm{Re} f(x) \chi(x) > 0$$

for some continuous (complex) character χ of G, then g is continuous.

L. Szekelyhidi

---

8 Let $p \geq 1$ be a real number and $\mathbb{R}_+ = (0, \infty)$. For which continuous functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ are following functions all convex?

$$M_n(x) = \left[\frac{\sum_{i=1}^{n} g(\frac{x_i}{x_{i+1}}) x_{i+1}^p}{\sum_{i=1}^{n} g(\frac{x_i}{x_{i+1}})} \right]^{\frac{1}{p}},$$

$$x = (x_1, \dots, x_{n+1}) \in \mathbb{R}_+^{n+1}, \ n = 1, 2, \dots$$

L. Losonczi

---

9 Suppose that the components of he vector **u** $= (u_0, \dots, u_n)$ are real functions defined on the closed interval $[a,b]$ with the property that every nontrivial linear combination of them has at most n zeros in $[a,b]$. Prove that if σ is an increasing function on $[a,b]$ and the rank of the operator

$$A(f) = \int_a^b \mathbf{u}(x) f(x) d\sigma(x), \ f \in C[a,b] \ ,$$

is $r \leq n$, then σ has exactly r points of increase.

E. Gesztelyi

---

10 Let the sequence of random variables $\{X_m,\ m \ge 0\},\ X_0 = 0$, be an infinite random walk on the set of nonnegative integers with transition probabilities

$$p_i = P(X_{m+1} = i+1 \mid X_m = i) > 0,\ i \ge 0$$

$$q_i = P(X_{m+1} = i-1 \mid X_m = i) > 0,\ i > 0.$$

Prove that for arbitrary $k > 0$ there is an $\alpha_k > 1$ such that

$$P_n(k) = P\left(\max_{0 \le j \le n} X_j = k\right)$$

satisfies the limit relation

$$\lim_{L \to \infty} \frac{1}{L} \sum_{n=1}^{L} P_n(k) \alpha_k^n < \infty.$$

J. Tomko