Mikls Schweitzer 1978

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by ehsan2004

1 Let $\mathcal{H}$ be a family of finite subsets of an infinite set X such that every finite subset of X can be represented as the union of two disjoint sets from $\mathcal{H}$. Prove that for every positive integer k there is a subset of X that can be represented in at least k different ways as the union of two disjoint sets from $\mathcal{H}$.

P. Erdos

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2 For a distributive lattice L, consider the following two statements:

(A) Every ideal of L is the kernel of at least two different homomorphisms.

(B) L contains no maximal ideal.

Which one of these statements implies the other?

(Every homomorphism φ of L induces an equivalence relation on L: $a \sim b$ if and only if $a\varphi = b\varphi$. We do not consider two homomorphisms different if they imply the same equivalence relation.)

J. Varlet, E. Fried

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3 Let $1 < a_1 < a_2 < \ldots < a_n < x$ be positive integers such that $\sum_{i=1}^n 1/a_i \le 1$. Let y denote the number of positive integers smaller that x not divisible by any of the a_i. Prove that

$$y > \frac{cx}{\log x}$$

with a suitable positive constant c (independent of x and the numbers a_i).

I. Z. Ruzsa

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4 Let $\mathbb{Q}$ and $\mathbb{R}$ be the set of rational numbers and the set of real numbers, respectively, and let $f : \mathbb{Q} \to \mathbb{R}$ be a function with the following property. For every $h \in \mathbb{Q},\ x_0 \in \mathbb{R}$,

$$f(x+h) - f(x) \to 0$$

as $x \in \mathbb{Q}$ tends to x_0. Does it follow that f is bounded on some interval?

M. Laczkovich

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5 Suppose that $R(z)=\sum_{n=-\infty}^{\infty} a_n z^n$ converges in a neighborhood of the unit circle $\{z:\ |z|=1\}$ in the complex plane, and $R(z)=P(z)/Q(z)$ is a rational function in this neighborhood, where P and Q are polynomials of degree at most k. Prove that there is a constant c independent of k such that

$$\sum_{n=-\infty}^{\infty} |a_n| \le ck^2 \max_{|z|=1} |R(z)|.$$

H. S. Shapiro, G. Somorjai

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6 Suppose that the function $g:(0,1)\to\mathbb{R}$ can be uniformly approximated by polynomials with nonnegative coefficients. Prove that g must be analytic. Is the statement also true for the interval $(-1,0)$ instead of $(0,1)$?

J. Kalina, L. Lempert

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7 Let T be a surjective mapping of the hyperbolic plane onto itself which maps collinear points into collinear points. Prove that T must be an isometry.

M. Bognar

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8 Let $X_1,\dots,X_n$ be n points in the unit square ($n>1$). Let r_i be the distance of X_i from the nearest point (other than X_i). Prove that the inequality

$$r_1^2+\ldots+r_n^2\le 4.$$

L. Fejes-Toth, E. Szemeredi

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9 Suppose that all subspaces of cardinality at most $\aleph_1$ of a topological space are second-countable. Prove that the whole space is second-countable.

A. Hajnal, I. Juhasz

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10 Let Y_n be a binomial random variable with parameters n and p. Assume that a certain set H of positive integers has a density and that this density is equal to d. Prove the following statements:

(a) $\lim_{n\to\infty} P(Y_n\in H)=d$ if H is an arithmetic progression.

(b) The previous limit relation is not valid for arbitrary H.

(c) If H is such that $P(Y_n\in H)$ is convergent, then the limit must be equal to d.

L. Posa

