Mikls Schweitzer 1980

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by ehsan2004

1 For a real number x, let $\|x\|$ denote the distance between x and the closest integer. Let $0 \leq x_n < 1 \; (n=1,2,\ldots)$, and let $\varepsilon > 0$. Show that there exist infinitely many pairs (n,m) of indices such that $n \neq m$ and

$$\|x_n - x_m\| < \min \left(\varepsilon, \frac{1}{2|n-m|} \right).$$

V. T. Sos

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2 Let $\mathcal{H}$ be the class of all graphs with at most $2^{\aleph_0}$ vertices not containing a complete subgraph of size $\aleph_1$. Show that there is no graph $H \in \mathcal{H}$ such that every graph in $\mathcal{H}$ is a subgraph of H.

F. Galvin

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3 In a lattice, connected the elements $a \wedge b$ and $a \vee b$ by an edge whenever a and b are incomparable. Prove that in the obtained graph every connected component is a sublattice.

M. Ajtai

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4 Let $T \in SL(n,\mathbb{Z})$, let G be a nonsingular $n \times n$ matrix with integer elements, and put $S=G^{-1}TG$. Prove that there is a natural number k such that $S^k \in SL(n,\mathbb{Z})$.

Gy. Szekeres

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5 Let G be a transitive subgroup of the symmetric group S_{25} different from S_{25} and A_{25}. Prove that the order of G is not divisible by 23.

J. Pelikan

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6 Let us call a continuous function $f : [a,b] \rightarrow \mathbb{R}^2$ *reducible* if it has a double arc (that is, if there are $a \leq \alpha < \beta \leq \gamma < \delta \leq b$ such that there exists a strictly monotone and continuous $h : [\alpha,\beta] \rightarrow [\gamma,\delta]$ for which $f(t)=f(h(t))$ is satisfied for every $\alpha \leq t \leq \beta$); otherwise f is irreducible. Construct irreducible $f : [a,b] \rightarrow \mathbb{R}^2$ and $g : [c,d] \rightarrow \mathbb{R}^2$ such that $f([a,b])=g([c,d])$ and

(a) both f and g are rectifiable but their lengths are different;

(b) f is rectifiable but g is not.

A. Csaszar

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7 Let $n \geq 2$ be a natural number and $p(x)$ a real polynomial of degree at most n for which

$$\max_{-1 \leq x \leq 1} |p(x)| \leq 1,\ p(-1) = p(1) = 0\,.$$

Prove that then

$$|p'(x)| \leq \frac{n \cos \frac{\pi}{2n}}{\sqrt{1 - x^2 \cos^2 \frac{\pi}{2n}}} \qquad \left(-\frac{1}{\cos \frac{\pi}{2n}} < x < \frac{1}{\cos \frac{\pi}{2n}} \right).$$

J. Szabados

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8 Let $f(x)$ be a nonnegative, integrable function on $(0, 2\pi)$ whose Fourier series is $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(n_k x)$, where none of the positive integers n_k divides another. Prove that $|a_k| \leq a_0$.

G. Halasz

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9 Let us divide by straight lines a quadrangle of unit area into n subpolygons and draw a circle into each subpolygon. Show that the sum of the perimeters of the circles is at most $\pi\sqrt{n}$ (the lines are not allowed to cut the interior of a subpolygon).

G. and L. Fejes-Toth

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10 Suppose that the T_3-space X has no isolated points and that in X any family of pairwise disjoint, nonempty, open sets
is countable. Prove that X can be covered by at most continuum many nowhere-dense sets.

I. Juhasz
