Mikls Schweitzer 1981

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by ehsan2004

1 We are given an infinite sequence of 1's and 2's with the following properties:

(1) The first element of the sequence is 1.

(2) There are no two consecutive 2's or three consecutive 1's.

(3) If we replace consecutive 1's by a single 2, leave the single 1's alone, and delete the original 2's, then we recover the original sequence.

How many 2's are there among the first n elements of the sequence?

P. P. Palfy

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2 Consider the lattice L of the contradictions of a simple graph G (as sets of vertex pairs) with respect to inclusion. Let $n \geq 1$ be an arbitrary integer. Show that the identity

$$x \bigwedge \left(\bigvee_{i=0}^{n} y_i \right) = \bigvee_{j=0}^{n} \left(x \bigwedge \left(\bigvee_{0 \leq i \leq n, \ i \neq j} y_i \right) \right)$$

holds if and only if G has no cycle of size at least $n+2$.

A. Huhn

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3 Construct an uncountable Hausdorff space in which the complement of the closure of any nonempty, open set is countable.

A. Hajnal, I. Juhasz

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4 Let G be finite group and $\mathcal{K}$ a conjugacy class of G that generates G. Prove that the following two statements are equivalent:

(1) There exists a positive integer m such that every element of G can be written as a product of m (not necessarily distinct) elements of $\mathcal{K}$.

(2) G is equal to its own commutator subgroup.

J. Denes

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5 Let K be a convex cone in the n-dimensional real vector space $\mathbb{R}^n$, and consider the sets $A = K \cup (-K)$ and $B = (\mathbb{R}^n \setminus A) \cup \{0\}$ (0 is the origin). Show that one can find two subspaces in $\mathbb{R}^n$ such that together they span $\mathbb{R}^n$, and one of them lies in A and the other lies in B.

J. Szucs

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6 Let f be a strictly increasing, continuous function mapping $I = [0,1]$ onto itself. Prove that the following inequality holds for all pairs $x, y \in I$:

$$1 - \cos(xy) \le \int_0^x f(t)\sin(tf(t))dt + \int_0^y f^{-1}(t)\sin(tf^{-1}(t))dt.$$

Zs. Pales

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7 Let U be a real normed space such that, for an finite-dimensional, real normed space X, U contains a subspace isometrically isomorphic to X. Prove that every (not necessarily closed) subspace V of U of finite codimension has the same property. (We call V of finite codimension if there exists a finite-dimensional subspace N of U such that $V + N = U$.)

A. Bosznay

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8 Let W be a dense, open subset of the real line $\mathbb{R}$. Show that the following two statements are equivalent:

(1) Every function $f : \mathbb{R} \to \mathbb{R}$ continuous at all points of $\mathbb{R} \setminus W$ and nondecreasing on every open interval contained in W is nondecreasing on the whole $\mathbb{R}$.

(2) $\mathbb{R} \setminus W$ is countable.

E. Gesztelyi

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9 Let $n \ge 2$ be an integer, and let X be a connected Hausdorff space such that every point of X has a neighborhood homeomorphic to the Euclidean space $\mathbb{R}^n$. Suppose that any discrete (not necessarily closed) subspace D of X can be covered by a family of pairwise disjoint, open sets of X so that each of these open sets contains precisely one element of D. Prove that X is a union of at most $\aleph_1$ compact subspaces.

Z. Balogh

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10 Let P be a probability distribution defined on the Borel sets of the real line. Suppose that P is symmetric with respect to the origin, absolutely continuous with respect to the Lebesgue measure, and its density function p is zero outside the interval $[-1,1]$ and inside this interval it is between the positive numbers c and d ($c < d$). Prove that there is no distribution whose convolution square equals P.

T. F. Mori, G. J. Szekely