Mikls Schweitzer 1982

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by ehsan2004

1 A map $F : P(X) \to P(X)$, where $P(X)$ denotes the set of all subsets of X, is called a *closure operation* on X if for arbitrary $A, B \subset X$, the following conditions hold:

(i) $A \subset F(A)$;

(ii) $A \subset B \Rightarrow F(A) \subset F(B)$;

(iii) $F(F(A)) = F(A)$.

The cardinal number $\min\{|A| : \ A \subset X \ , \ F(A) = X \ \}$ is called the *density* of F and is denoted by $d(F)$. A set $H \subset X$ is called *discrete* with respect to F if $u \notin F(H - \{u\})$ holds for all $u \in H$. Prove that if the density of the closure operation F is a singular cardinal number, then for any nonnegative integer n, there exists a set of size n that is discrete with respect to F. Show that the statement is not true when the existence of an infinite discrete subset is required, even if F is the closure operation of a topological space satisfying the T_1 separation axiom.

A. Hajnal

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2 Consider the lattice of all algebraically closed subfields of the complex field $\mathbb{C}$ whose transcendency degree (over $\mathbb{Q}$) is finite. Prove that this lattice is not modular.

L. Babai

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3 Let $G(V,E)$ be a connected graph, and let $d_G(x,y)$ denote the length of the shortest path joining x and y in G. Let $r_G(x) = \max\{d_G(x,y) : \ y \in V \ \}$ for $x \in V$, and let $r(G) = \min\{r_G(x) : \ x \in V \ \}$. Show that if $r(G) \geq 2$, then G contains a path of length $2r(G)-2$ as an induced subgraph.

V. T. Sos

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4 Let

$$f(n) = \sum_{p|n, \ p^{\alpha} \leq n < p^{\alpha+1}} p^{\alpha}.$$

Prove that

$$\limsup_{n \to \infty} f(n) \frac{\log \log n}{n \log n} = 1.$$

P. Erdos

5 Find a perfect set $H \subset [0,1]$ of positive measure and a continuous function f defined on $[0,1]$ such that for any twice differentiable function g defined on $[0,1]$, the set $\{x \in H : \ f(x) = g(x)\}$ is finite.

M. Laczkovich

6 For every positive α, natural number n, and at most αn points x_i, construct a trigonometric polynomial $P(x)$ of degree at most n for which

$$P(x_i) \leq 1, \ \int_0^{2\pi} P(x)dx = 0, \ \text{ and } \ \max P(x) > cn \ ,$$

where the constant c depends only on α.

G. Halasz

7 Let V be a bounded, closed, convex set in $\mathbb{R}^n$, and denote by r the radius of its circumscribed sphere (that is, the radius of the smallest sphere that contains V). Show that r is the only real number with the following property: for any finite number of points in V, there exists a point in V such that the arithmetic mean of its distances from the other points is equal to r.

Gy. Szekeres

8 Show that for any natural number n and any real number $d > 3^n/(3^n - 1)$, one can find a covering of the unit square with n homothetic triangles with area of the union less than d.

9 Suppose that K is a compact Hausdorff space and $K = \cup_{n=0}^{\infty} A_n$, where A_n is metrizable and $A_n \subset A_m$ for $n < m$. Prove that K is metrizable.

Z. Balogh

10 Let $p_0, p_1, \ldots$ be a probability distribution on the set of nonnegative integers. Select a number according to this distribution and repeat the selection independently until either a zero or an already selected number is obtained. Write the selected numbers in a row in order of selection without the last one. Below this line, write the numbers again in increasing order. Let A_i denote the event that the number i has been selected and that it is in the same place in both lines. Prove that the events A_i $(i = 1, 2, \ldots)$ are mutually independent, and $P(A_i) = p_i$.

T. F. Mori