Mikls Schweitzer 1983

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by ehsan2004

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1 Given n points in a line so that any distance occurs at most twice, show that the number of distance occurring exactly once is at least $\lfloor n/2 \rfloor$.

V. T. Sos, L. Szekely

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2 Let I be an ideal of the ring R and f a nonidentity permutation of the set $\{1, 2, \dots, k\}$ for some k. Suppose that for every $0 \neq a \in R,\ aI \neq 0$ and $Ia \neq 0$ hold; furthermore, for any elements $x_1, x_2, \dots, x_k \in I$,

$$x_1 x_2 \dots x_k = x_{1f} x_{2f} \dots x_{kf}$$

holds. Prove that R is commutative.

R. Wiegandt

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3 Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable, 2π-periodic even function. Prove that if

$$f''(x) + f(x) = \frac{1}{f(x + 3\pi/2)}$$

holds for every x, then f is $\pi/2$-periodic.

Z. Szabo, J. Terjeki

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4 For which cardinalities κ do antimetric spaces of cardinality κ exist? (X, ϱ) is called an *antimetric space* if X is a nonempty set, $\varrho : X^2 \to [0, \infty)$ is a symmetric map, $\varrho(x, y) = 0$ holds iff $x = y$, and for any three-element subset $\{a_1, a_2, a_3\}$ of X

$$\varrho(a_{1f}, a_{2f}) + \varrho(a_{2f}, a_{3f}) < \varrho(a_{1f}, a_{3f})$$

holds for some permutation f of $\{1, 2, 3\}$.

V. Totik

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5 Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $x + g(x)$ is strictly monotone (increasing or decreasing), and let $u : [0, \infty) \to \mathbb{R}$ be a bounded and continuous function such that

$$u(t) + \int_{t-1}^{t} g(u(s))ds$$

is constant on $[1, \infty)$. Prove that the limit $\lim_{t\to\infty} u(t)$ exists.

T. Krisztin

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6 Let T be a bounded linear operator on a Hilbert space H, and assume that $\|T^n\| \leq 1$ for some natural number n. Prove the existence of an invertible linear operator A on H such that $\|ATA^{-1}\| \leq 1$.

E. Druszt

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7 Prove that if the function $f : \mathbb{R}^2 \to [0,1]$ is continuous and its average on every circle of radius 1 equals the function value at the center of the circle, then f is constant.

V. Totik

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8 Prove that any identity that holds for every finite n-distributive lattice also holds for the lattice of all convex subsets of the $(n-1)$-dimensional Euclidean space. (For convex subsets, the lattice operations are the set-theoretic intersection and the convex hull of the set-theoretic union. We call a lattice n-*distributive* if

$$x \wedge (\bigvee_{i=0}^{n} y_i) = \bigvee_{j=0}^{n} (x \wedge (\bigvee_{0\leq i\leq n,\ i\neq j} y_i))$$

holds for all elements of the lattice.)

A. Huhn

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9 Prove that if $E \subset \mathbb{R}$ is a bounded set of positive Lebesgue measure, then for every $u < 1/2$, a point $x = x(u)$ can be found so that

$$|(x-h, x+h) \cap E| \geq uh$$

and

$$|(x-h, x+h) \cap (\mathbb{R} \setminus E)| \geq uh$$

for all sufficiently small positive values of h.

K. I. Koljada

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10 Let R be a bounded domain of area t in the plane, and let C be its center of gravity. Denoting by T_{AB} the circle drawn with the diameter AB, let K be a circle that contains each of the circles T_{AB} $(A, B \in R)$. Is it true in general that K contains the circle of area $2t$ centered at C?

J. Szucs

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11 Let $M^n \subset \mathbb{R}^{n+1}$ be a complete, connected hypersurface embedded into the Euclidean space. Show that M^n as a Riemannian manifold decomposes to a nontrivial global metric direct product if and only if it is a real cylinder, that is, M^n can be decomposed to a direct product of the form $M^n = M^k \times \mathbb{R}^{n-k}$ $(k < n)$ as well, where M^k is a hypersurface in some $(k+1)$-dimensional subspace $E^{k+1} \subset \mathbb{R}^{n+1}, \mathbb{R}^{n-k}$ is the orthogonal complement of E^{k+1}.

Z. Szabo

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12 Let $X_1, X_2, \dots, X_n$ be independent, identically distributed, nonnegative random variables with a common continuous distribution function F. Suppose in addition that the inverse of F, the quantile function Q, is also continuous and $Q(0) = 0$. Let $0 = X_{0:n} \leq X_{1:n} \leq \dots \leq X_{n:n}$ be the ordered sample from the above random variables. Prove that if EX_1 is finite, then the random variable

$$\Delta = \sup_{0 \leq y \leq 1} \left| \frac{1}{n} \sum_{i=1}^{\lfloor ny \rfloor + 1} (n+1-i)(X_{i:n} - X_{i-1:n}) - \int_0^y (1-u) dQ(u) \right|$$

tends to zero with probability one as $n \to \infty$.

S. Csorgp, L. Horvath
