Tournament Of Towns 1998

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–	Spring 1998
–	Juniors O Level
1	Anya, Borya, and Vasya listed words that could be formed from a given set of letters. They each listed a different number of words : Anya listed the most, Vasya the least . They were awarded points as follows. Each word listed by only one of them scored 2 points for this child. Each word listed by two of them scored 1 point for each of these two children. Words listed by all three of them scored 0 points. Is it possible that Vasya got the highest score, and Anya the lowest? (A Shapovalov)
2	A chess king tours an entire 8×8 chess board, visiting each square exactly once and returning at last to his starting position. Prove that he made an even number of diagonal moves. (V Proizvolov)
3	AB and CD are segments lying on the two sides of an angle whose vertex is O. A is between O and B, and C is between O and D . The line connecting the midpoints of the segments AD and BC intersects AB at M and CD at N. Prove that $\frac{OM}{ON} = \frac{AB}{CD}$ (V Senderov)
4	For every three-digit number, we take the product of its three digits. Then we add all of these products together. What is the result? (G Galperin)
5	Pinocchio claims that he can divide an isoceles triangle into three triangles, any two of which can be put together to form a new isosceles triangle. Is Pinocchio lying? (A Shapovalov)
–	Juniors A Level
1	Do there exist 10 positive integers such that each of them is divisible by none of the other numbers but the square of each of these numbers is divisible by each of the other numbers? (Folklore)

2 $ABCD$ is a parallelogram. A point M is found on the side AB or its extension such that $\angle MAD = \angle AMO$ where O is the intersection point of the diagonals of the parallelogram. Prove that $MD = MG$.

(M Smurov)

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3 Six dice are strung on a rigid wire so that the wire passes through two opposite faces of each die. Each die can be rotated independently of the others. Prove that it is always possible to rotate the dice and then place the wire horizontally on a table so that the six-digit number formed by their top faces is divisible by 7. (The faces of a die are numbered from 1 to 6, the sum of the numbers on opposite faces is always equal to 7.)

(G Galperin)

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4 A traveller visited a village whose inhabitants either always tell the truth or always lie. The villagers stood in a circle facing the centre of the circle, and each villager announced whether the person standing to his right is a truth-teller. On the basis of this information, the traveller was able to determine what fraction of the villagers were liars. What was this fraction?

(B, Frenkin)

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5 A square is divided into 25 small squares. We draw diagonals of some of the small squares so that no two diagonals share a common point (not even a common endpoint). What is the largest possible number of diagonals that we can draw?

(I Rubanov)

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6 10 people are sitting at a round table. There are some nuts in front of each of them, 100 nuts altogether. After a certain signal each person passes some of his nuts to the person sitting to his right . If he has an even number of nuts, he passes half of them; otherwise he passes one nut plus half of the remaining nuts. This procedure is repeated over and over again. Prove that eventually everyone will have exactly 10 nuts.

(A Shapovalov)

– Seniors O Level

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1 Pinocchio claims that he can take some non-right-angled triangles , all of which are similar to one another and some of which may be congruent to one another, and put them together to form a rectangle. Is Pinocchio lying?

(A Fedotov)

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2 For every four-digit number, we take the product of its four digits. Then we add all of these products together . What is the result?

(G Galperin)

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3 What is the maximum number of colours that can be used to paint an 8×8 chessboard so that every square is painted in a single colour, and is adjacent , horizontally, vertically but not diagonally, to at least two other squares of its own colour?

(A Shapovalov)

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4 For some positive numbers A, B, C and D, the system of equations

$x^2 + y^2 = A$ and $|x| + |y| = B$

has m solutions, while the system of equations

$x^2 + y^2 + z^2 = X$ and $|x| + |y| + |z| = D$

has n solutions. If $m > n > 1$, find m and n.

(G Galperin)

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5 A circle with center O is inscribed in an angle. Let A be the reflection of O across one side of the angle. Tangents to the circle from A intersect the other side of the angle at points B and C. Prove that the circumcenter of triangle ABC lies on the bisector of the original angle.

(I.Sharygin)

– Seniors A Level

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1 Prove that

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}$$

for positive reals a, b, c

(S Tokarev)

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2 A square of side 1 is divided into rectangles . We choose one of the two smaller sides of each rectangle (if the rectangle is a square, then we choose any of the four sides) . Prove that the sum of the lengths of all the chosen sides is at least 1 .

(Folklore)

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3 (a) The numbers $1, 2, 4, 8, 16, 32, 64, 128$ are written on a blackboard.
We are allowed to erase any two numbers and write their difference instead (this is always a non-negative number). After this procedure has been repeated seven times, only a single number will remain. Could this number be 97?

(b) The numbers $1, 2, 22, 23, ..., 210$ are written on a blackboard.

We are allowed to erase any two numbers and write their difference instead (this is always a non-negative number) . After this procedure has been repeated ten times, only a single number will remain. What values could this number have?

(A.Shapovalov)

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4 A point M is found inside a convex quadrilateral $ABCD$ such that triangles AMB and CMD are isoceles ($AM = MB, CM = MD$) and $\angle AMB = \angle CMD = 120^o$. Prove that there exists a point N such that trianglesBNC and DNA are equilateral.

(I.Sharygin)

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5 A "labyrinth" is an 8×8 chessboard with walls between some neighboring squares. If a rook can traverse the entire board without jumping over the walls, the labyrinth is "good" ; otherwise it is "bad" . Are there more good labyrinths or bad labyrinths?

(A Shapovalov)

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6 (a) Two people perform a card trick. The first performer takes 5 cards from a 52-card deck (previously shuffled by a member of the audience) , looks at them, and arranges them in a row from left to right: one face down (not necessarily the first one) , the others face up . The second performer guesses correctly the card which is face down. Prove that the performers can agree on a system which always makes this possible.
(b) For their second trick, the first performer arranges four cards in a row, face up, the fifth card is kept hidden. Can they still agree on a system which enables the second performer to correctly guess the hidden card?

(G Galperin)

– Autumn 1998

– Juniors O Level

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1 A $20 \times 20 \times 20$ block is cut up into 8000 non-overlapping unit cubes and a number is assigned to each. It is known that in each column of 20 cubes parallel to any edge of the block, the sum of their numbers is equal to 1. The number assigned to one of the unit cubes is 10. Three $1 \times 20 \times 20$ slices parallel to the faces of the block contain this unit cube. Find the sume of all numbers of the cubes outside these slices.

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2 The units-digit of the square of an integer is 9 and the tens-digit of this square is 0. Prove that the hundreds-digit is even.

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3 In a triangle ABC the points A', B' and C' lie on the sides BC, CA and AB, respectively. It is known that $\angle AC'B' = \angle B'A'C$, $\angle CB'A' = \angle A'C'B$ and $\angle BA'C' = \angle C'B'A$. Prove that A', B' and C' are the midpoints of the corresponding sides.

4 Twelve candidates for mayor participate in a TV talk show. At some point a candidate said: "One lie has been told." Another said: "Now two lies have been told". "Now three lies," said a third. This continued until the twelfth said: "Now twelve lies have been told". At this point the moderator ended the discussion. It turned out that at least one of the candidates correctly stated the number of lies told before he made the claim. How many lies were actually told by the candidates?

5 Let n and m be given positive integers. In one move, a chess piece called an (n, m)-crocodile goes n squares horizontally or vertically and then goes m squares in a perpendicular direction. Prove that the squares of an infinite chessboard can be painted in black and white so that this chess piece always moves from a black square to a white one or vice-versa.

– Juniors A Level

1 part a of Seniors A p1

2 John and Mary each have a white 8×8 square divided into 1×1 cells. They have painted an equal number of cells on their respective squares in blue. Prove that one can cut up each of the two squares into 2×1 dominoes so that it is possible to reassemble John's dominoes into a new square and Mary's dominoes into another square with the same pattern of blue cells.

(A Shapovalov)

3 Segment AB intersects two equal circles, is parallel to the line joining their centres, and all the points of intersection of the segment and the circles lie between A and B. From the point A tangents to the circle nearest to A are drawn, and from the point B tangents to the circle nearest to B are also drawn. It turns out that the quadrilateral formed by the four tangents extended contains both circles. Prove that a circle can be drawn so that it touches all four sides of the quadrilateral.

(P Kozhevnikov)

4 All the diagonals of a regular 25-gon are drawn. Prove that no 9 of the diagonals pass through one interior point of the 25-gon.

(A Shapovalov)

5 There are 20 beads of 10 colours, two of each colour. They are put in 10 boxes. It is known that one bead can be selected from each of the boxes so that each colour is represented. Prove that the number of such selections is a non-zero power of 2.

(A Grishin)

6 A gang of robbers took away a bag of coins from a merchant . Each coin is worth an integer number of pennies. It is known that if any single coin is removed from the bag, then the remaining coins can be divided fairly among the robbers (that is, they all get coins with the same total value in pennies) . Prove that after one coin is removed, the number of the remaining coins is divisible by the number of robbers.

(Folklore, modified by A Shapovalov)

– Seniors O Level

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1 Nineteen weights of mass 1 gm, 2 gm, 3 gm, . . . , 19 gm are given. Nine are made of iron, nine are of bronze and one is pure gold. It is known that the total mass of all the iron weights is 90 gm more than the total mass of all the bronze ones. Find the mass of the gold weight .

(V Proizvolov)

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2 On the plane are n paper disks of radius 1 whose boundaries all pass through a certain point, which lies inside the region covered by the disks. Find the perimeter of this region.

(P Kozhevnikov)

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3 On an 8×8 chessboard, 17 cells are marked. Prove that one can always choose two cells among the marked ones so that a Knight will need at least three moves to go from one of the chosen cells to the other.

(R Zhenodarov)

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4 Among all sets of real numbers $\{x_1,x_2,...,x_{20}\}$ from the open interval $(0,1)$ such that $x_1x_2...x_{20} = (1-x_1)(1-x_2)...(1-x_{20})$, find the one for which $x_1x_2...x20$ is maximal.

(A Cherniatiev)

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5 The intelligence quotient (IQ) of a country is defined as the average IQ of its entire population. It is assumed that the total population and individual IQs remain constant throughout.
(a) (i) A group of people from country A has emigrated to country B . Show that it can happen that as a result , the IQs of both countries have increased.
(ii) After this, a group of people from B, which may include immigrants from A, emigrates to A. Can it happen that the IQs of both countries will increase again?
(b) A group of people from country A has emigrated to country B, and a group of people from B has emigrated to country C . It is known that a s a result , the IQs o f all three countries have increased. After this, a group of people from C emigrates to B and a group of people from B emigrates to A. Can it happen that the IQs of all three countries will increase again?

(A Kanel, B Begun)

– Seniors A Level

1 (a) Prove that for any two positive integers a and b the equation $lcm(a, a + 5) = lcm(b, b + 5)$ implies $a = b$.
(b) Is it possible that $lcm(a, b) = lcm(a + c, b + c)$ for positive integers a, b and c?

(A Shapovalov)

PS. part (a) for Juniors, both part for Seniors

2 same as Juniors A p3

3 Nine numbers are arranged in a square table: $a_1\ \ a_2\ \ a_3\ b_1\ \ b_2\ \ b_3\ c_1\ \ c_2\ \ c_3$.
It is known that the six numbers obtained by summing the rows and columns of the table are equal: $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = c_1 + c_2 + c_3 = a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3$.
Prove that the sum of products of numbers in the rows is equal to the sum of products of numbers in the columns: $a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 = a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3$.

(V Proizvolov)

4 Twelve places have been arranged at a round table for members of the Jury, with a name tag at each place . Professor K. being absent-minded instead of occupying his place, sits down at the next place (clockwise) . Each of the other Jury members in turn either occupies the place assigned to this member or, if it has been already occupied, sits down at the first free place in the clockwise order. The resulting seating arrangement depends on the order in which the Jury members come to the table. How many different seating arrangements of this kind are possible?

(A Shapovalov)

5 The sum of the length, width, and height of a rectangular parallelepiped will be called its size. Can it happen that one rectangular parallelepiped contains another one of greater size?

(A Shen)

6 In a function $f(x) = (x_2 + ax + b)/(x^2 + cx + d)$, the quadratics $x^2 + ax + b$ and $x^2 + cx + d$ have no common roots. Prove that the next two statements are equivalent:
(i) there is a numerical interval without any values of $f(x)$,
(ii) $f(x)$ can be represented in the form $f(x) = f_1(f_2(...f_{n-1}(f_n(x))...))$ where each of the functions f_j is o f one of the three forms $k_jx + b_j, 1/x, x^2$.

(A Kanel)