AoPS Online
Art of Problem Solving

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Day 1 November 23rd

1 The sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_0 = u_1 = 1$,$u_n = 2u_{n-1} - 3u_{n-2}$ $(n \geq 2)$, $v_0 = a, v_1 = b, v_2 = c$,$v_n = v_{n-1} - 3v_{n-2} + 27v_{n-3}$ $(n \geq 3)$. There exists a positive integer N such that when $n > N$, we have $u_n \mid v_n$. Prove that $3a = 2b + c$.

2 In acute triangle ABC, let $\odot O$ be its circumcircle, $\odot I$ be its incircle. Tangents at B, C to $\odot O$ meet at L, $\odot I$ touches BC at D. AY is perpendicular to BC at Y, AO meets BC at X, and OI meets $\odot O$ at P, Q. Prove that P, Q, X, Y are concyclic if and only if A, D, L are collinear.

3 Consider a rectangle R partitioned into 2016 smaller rectangles such that the sides of each smaller rectangle is parallel to one of the sides of the original rectangle. Call the corners of each rectangle a vertex. For any segment joining two vertices, call it basic if no other vertex lie on it. (The segments must be part of the partitioning.) Find the maximum/minimum possible number of basic segments over all possible partitions of R.

Day 2 November 24th

4 Let $n \geq 2$ be a natural number. For any two permutations of $(1, 2, \cdots, n)$, say $\alpha = (a_1, a_2, \cdots, a_n)$ and $\beta = (b_1, b_2, \cdots, b_n)$, if there exists a natural number $k \leq n$ such that

$$b_i = \begin{cases} a_{k+1-i}, & 1 \leq i \leq k; \\ a_i, & k < i \leq n, \end{cases}$$

we call α a friendly permutation of β.

Prove that it is possible to enumerate all possible permutations of $(1, 2, \cdots, n)$ as $P_1, P_2, \cdots, P_m$ such that for all $i = 1, 2, \cdots, m$, P_{i+1} is a friendly permutation of P_i where $m = n!$ and $P_{m+1} = P_1$.

5 Let D_n be the set of divisors of n. Find all natural n such that it is possible to split D_n into two disjoint sets A and G, both containing at least three elements each, such that the elements in A form an arithmetic progression while the elements in G form a geometric progression.

6 Given an integer $n \geq 2$ and real numbers a, b such that $0 < a < b$. Let $x_1, x_2, \dots, x_n \in [a, b]$ be real numbers. Find the maximum value of

$$\frac{\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1}}{x_1 + x_2 + \cdots + x_{n-1} + x_n}.$$