Tournament Of Towns 2002

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by joybangla

–   I. Spring - Junior O-Level

1   There are many $a \times b$ rectangular cardboard pieces ($a, b \in \mathbb{N}$ such that $a < b$). It is given that by putting such pieces together without overlapping one can make 49×51 rectangle, and 99×101 rectangle. Can one uniquely determine a, b from this?

2   Can any triangle be cut into four convex figures: a triangle, a quadrilateral, a pentagon, a hexagon?

3   Show that if the last digit of the number $x^2 + xy + y^2$ is 0 (where $x, y \in \mathbb{N}$) then last two digits are zero.

4   Quadrilateral $ABCD$ is circumscribed about a circle Γ and K, L, M, N are points of tangency of sides AB, BC, CD, DA with Γ respectively. Let $S \equiv KM \cap LN$. If quadrilateral $SKBL$ is cyclic then show that $SNDM$ is also cyclic.

5

    - There are 128 coins of two different weights, 64 each. How can one always find two coins of different weights by performing no more than 7 weightings on a regular balance?

    - There are 8 coins of two different weights, 4 each. How can one always find two coins of different weights by performing two weightings on a regular balance?

–   I. Spring - Junior A-Level

1   Let a, b, c be sides of a triangle. Show that $a^3 + b^3 + 3abc > c^3$.

2   A game is played on a 23×23 board. The first player controls two white chips which start in the bottom left and top right corners. The second player controls two black ones which start in bottom right and top left corners. The players move alternately. In each move, a player moves one of the chips under control to a square which shares a side with the square the chip is currently in. The first player wins if he can bring the white chips to squares which share a side with each other. Can the second player prevent the first player from winning?

3   Let E and F be the respective midpoints of BC, CD of a convex quadrilateral $ABCD$. Segments AE, AF, EF cut the quadrilateral into four triangles whose areas are four consecutive integers. Find the maximum possible area of ΔBAD.

4   There are n lamps in a row. Some of which are on. Every minute all the lamps already on go off. Those which were off and were adjacent to exactly one lamp which was on will go on. For which n one can find an initial configuration of lamps which were on, such that at least one lamp will be on at any time?

5   An acute triangle was dissected by a straight cut into two pieces which are not necessarily triangles. Then one of the pieces were dissected by a straight cut into two pieces and so on. After a few dissections it turns out the pieces were all triangles. Is it possible they were all obtuse?

6   In an infinite increasing sequence of positive integers, every term from the 2002th term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.

7   Some domino pieces are placed in a chain according to standard rules. In each move, we may remove a sub-chain with equal numbers at its ends, turn the whole sub-chain around, and put it back in the same place. Prove that for every two legal chains formed from the same pieces and having the same numbers at their ends, we can transform one to another in a finite sequence of moves.

–   I. Spring - Senior - O-Level

1   Show that if the last digit of the number $x^2 + xy + y^2$ is 0 (where $x, y \in \mathbb{N}$) then last two digits are zero.

2 $\triangle ABC$ and its mirror reflection $\triangle A'B'C'$ is arbitrarily placed on the plane. Prove the midpoints of AA', BB', CC' are collinear.

3   There are 6 pieces of cheese of different weights. For any two pieces we can identify the heavier piece. Given that it is possible to divide them into two groups of equal weights with three pieces in each. Give the explicit way to find these groups by performing two weightings on a regular balance.

4   In how many ways can we place the numbers from 1 to 100 in a 2×50 rectangle (divided into 100 unit squares) so that any two consecutive numbers are always placed in squares with a common side?

5   Does there exist a regular triangular prism that can be covered (without overlapping) by different equilateral triangles? (One is allowed to bend the triangles around the edges of the prism.)

–   I. Spring - Senior - A-Level

1   In a triangle ABC it is given $\tan A, \tan B, \tan C$ are integers. Find their values.

2   Does there exist points A, B on the curve $y = x^3$ and on $y = x^3 + |x| + 1$ respectively such that distance between A, B is less than $\frac{1}{100}$?

3   In an infinite increasing sequence of positive integers, every term from the 2002th term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.

4   The spectators are seated in a row with no empty places. Each is in a seat which does not match the spectator's ticket. An usher can order two spectators in adjacent seats to trade places unless one of them is already seated correctly. Is it true that from any initial arrangement, the spectators can be brought to their correct seats?

5   Let AA_1, BB_1, CC_1 be the altitudes of acute $\triangle ABC$. Let O_a, O_b, O_c be the incentres of $\triangle AB_1C_1, \triangle BC_1A_1, \triangle$ respectively. Also let T_a, T_b, T_c be the points of tangency of the incircle of $\triangle ABC$ with BC, CA, AB respectively. Prove that $T_aO_cT_bO_aT_cO_b$ is an equilateral hexagon.

6   The 52 cards of a standard deck are placed in a 13×4 array. If every two adjacent cards, vertically or horizontally, have the same suit or have the same value, prove that all 13 cards of the same suit are in the same row.

7   Do there exist irrational numbers a, b both greater than 1, such that $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for all $m, n \in \mathbb{N}$?

–   II. Fall - Junior - O-Level

1   In a convex 2002-gon several diagonals are drawn so that they do not intersect inside the polygon. As a result the polygon splits into 2000 triangles. Isit possible that exactly 1000 triangles have diagonals for all their three sides?

2   John and Mary select a natural number each and tell that to Bill. Bill wrote their sum and product in two papers hid one paper and showed the other to John and Mary.
John looked at the number (which was 2002) and declared he couldn't determine Mary's number. Knowing this Mary also said she couldn't determine John's number as well.
What was Mary's Number?

3
- A test was conducted in class. It is known that at least $\frac{2}{3}$ of the problems were hard. Each such problems were not solved by at least $\frac{2}{3}$ of the students. It is also known that at least $\frac{2}{3}$ of the students passed the test. Each such student solved at least $\frac{2}{3}$ of the suggested problems. Is this possible?

- Previous problem with $\frac{2}{3}$ replaced by $\frac{3}{4}$.
- Previous problem with $\frac{2}{3}$ replaced by $\frac{7}{10}$.

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4 2002 cards with numbers $1, 2, \ldots, 2002$ written on them are put on a table face up. Two players A, B take turns to pick up a card until all are gone. A goes first. The player who gets the last digit of the sum of his cards larger than his opponent wins.
Who has a winning strategy and how should one play to win?

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5   An angle and a point A inside it is given. Is it possible to draw through A three straight lines so that on either side of the angle one of three points of intersection of these lines be the midpoint of two other points of intersection with that side?

–   II. Fall - Junior - A-Level

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1   There are 2002 employees in a bank. All the employees came to celebrate the bank's jubilee and were seated around one round table. It is known that the difference in salaries of any two adjacent employees is 2 or 3 dollars. Find the maximal difference in salaries of two employees, if it is known all salaries are different.

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2   All the species of plants existing in Russia are catalogued (numbered by integers from 2 to 2000 ; one after another, without omissions or repetitions). For any pair of species the gcd of their catalogue numbers was calculated and recorded but the catalogue numbers themselves were lost. Is it possible to restore the catalogue numbers from the data in hand?

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3   The vertices of a 50-gon divide a circumference into 50 arcs, whose lengths are $1, 2, \ldots 50$ in some order. It is known that any two opposite arcs (corresponding to opposite sides) differ by 25. Prove that the polygon has two parallel sides.

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4   Point P is chosen in the plane of triangle ABC such that $\angle ABP$ is congruent to $\angle ACP$ and $\angle CBP$ is congruent to $\angle CAP$. Show P is the orthocentre.

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5   A convex N-gon is divided by diagonals into triangles so that no two diagonals intersect inside the polygon. The triangles are painted in black and white so that any two triangles are painted in black and white so that any two triangles with a common side are painted in different colors. For each N find the maximal difference between the numbers of black and white triangles.

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6   There's a large pile of cards. On each card a number from $1, 2, \ldots n$ is written. It is known that sum of all numbers on all of the cards is equal to $k \cdot n!$ for some k. Prove that it is possible to arrange cards into k stacks so that sum of numbers written on the cards in each stack is equal to $n!$.

7
         - A power grid with the shape of a 3×3 lattice with 16 nodes (vertices of the lattice) joined by

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wires (along the sides of squares. It may have happened that some of the wires have burned out. In one test technician can choose any two nodes and check if electrical current circulates between them (i.e there is a chain of intact wires joining the chosen nodes) . Technicial knows that current will circulate from any node to another node. What is the least number of tests required to demonstrate this?
- Previous problem for the grid of 5×5 lattice.

–   II. Fall - Senior - O-Level

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1   John and Mary select a natural number each and tell that to Bill. Bill wrote their sum and product in two papers hid one paper and showed the other to John and Mary.
John looked at the number (which was 2002) and declared he couldn't determine Mary's number. Knowing this Mary also said she couldn't determine John's number as well.
What was Mary's Number?

2
- A test was conducted in class. It is known that at least $\frac{2}{3}$ of the problems were hard. Each such problems were not solved by at least $\frac{2}{3}$ of the students. It is also known that at least $\frac{2}{3}$ of the students passed the test. Each such student solved at least $\frac{2}{3}$ of the suggested problems. Is this possible?
- Previous problem with $\frac{2}{3}$ replaced by $\frac{3}{4}$.
- Previous problem with $\frac{2}{3}$ replaced by $\frac{7}{10}$.

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3   Several straight lines such that no two are parallel, cut the plane into several regions. A point A is marked inside of one region. Prove that a point, separated from A by each of these lines, exists if and only if A belongs to an unbounded region.

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4 $x, y, z \in \left(0, \frac{\pi}{2}\right)$ are given. Prove that:

$$\frac{x \cos x + y \cos y + z \cos z}{x + y + z} \leq \frac{\cos x + \cos y + \cos z}{3}$$

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5   An infinite sequence of natural number $\{x_n\}_{n \geq 1}$ is such that x_{n+1} is obtained by adding one of the non-zero digits of x_n to itself. Show this sequence contains an even number.

–   II. Fall - Senior - A-Level

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1   All the species of plants existing in Russia are catalogued (numbered by integers from 2 to 2000 ; one after another, without omissions or repetitions). For any pair of species the gcd of their catalogue numbers was calculated and recorded but the catalogue numbers themselves were lost. Is it possible to restore the catalogue numbers from the data in hand?

2   A cube is cut by a plane such that the cross section is a pentagon. Show there is a side of the pentagon of length ℓ such that the inequality holds:

$$|\ell - 1| > \frac{1}{5}$$

3   A convex N-gon is divided by diagonals into triangles so that no two diagonals intersect inside the polygon. The triangles are painted in black and white so that any two triangles are painted in black and white so that any two triangles with a common side are painted in different colors. For each N find the maximal difference between the numbers of black and white triangles.

4   There's a large pile of cards. On each card a number from $1, 2, \ldots n$ is written. It is known that sum of all numbers on all of the cards is equal to $k \cdot n!$ for some k. Prove that it is possible to arrange cards into k stacks so that sum of numbers written on the cards in each stack is equal to $n!$.

5   Two circles Γ_1, Γ_2 intersect at A, B. Through B a straight line ℓ is drawn and $\ell \cap \Gamma_1 = K, \ell \cap \Gamma_2 = M$ $(K, M \neq B)$. We are given $\ell_1 \parallel AM$ is tangent to Γ_1 at Q. $QA \cap \Gamma_2 = R$ $(\neq A)$ and further ℓ_2 is tangent to Γ_2 at R.
Prove that:

-$\ell_2 \parallel AK$
-ℓ, ℓ_1, ℓ_2 have a common point.

6   Define a sequence $\{a_n\}_{n \geq 1}$ such that $a_1 = 1, a_2 = 2$ and a_{n+1} is the smallest positive integer m such that m hasn't yet occurred in the sequence and also $\gcd(m, a_n) \neq 1$. Show all positive integers occur in the sequence.

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- A power grid with the shape of a 3×3 lattice with 16 nodes (vertices of the lattice) joined by wires (along the sides of squares. It may have happened that some of the wires have burned out. In one test technician can choose any two nodes and check if electrical current circulates between them (i.e there is a chain of intact wires joining the chosen nodes) . Technical knows that current will circulate from any node to another node. What is the least number of tests required to demonstrate this?
- Previous problem for the grid of 7×7 lattice.