Tournament Of Towns 2008

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–   Fall 2008

–   Junior O-Level

1   Each of ten boxes contains a di fferent number of pencils. No two pencils in the same box are of the same colour. Prove that one can choose one pencil from each box so that no two are of the same colour.

2   Twenty-fi ve of the numbers $1, 2, \cdots, 50$ are chosen. Twenty- five of the numbers $51, 52, \cdots, 100$ are also chosen. No two chosen numbers diff er by 0 or 50. Find the sum of all 50 chosen numbers.

3   Acute triangle $A_1A_2A_3$ is inscribed in a circle of radius 2. Prove that one can choose points B_1, B_2, B_3 on the arcs A_1A_2, A_2A_3, A_3A_1 respectively, such that the numerical value of the area of the hexagon $A_1B_1A_2B_2A_3B_3$ is equal to the numerical value of the perimeter of the triangle $A_1A_2A_3$.

4   Given three distinct positive integers such that one of them is the average of the two others. Can the product of these three integers be the perfect 2008th power of a positive integer?

5   On a straight track are several runners, each running at a di fferent constant speed. They start at one end of the track at the same time. When a runner reaches any end of the track, he immediately turns around and runs back with the same speed (then he reaches the other end and turns back again, and so on). Some time after the start, all runners meet at the same point. Prove that this will happen again.

–   Junior A-Level

1 100 Queens are placed on a 100×100 chessboard so that no two attack each other. Prove that each of four 50×50 corners of the board contains at least one Queen.

2   Each of 4 stones weights the integer number of grams. A balance with arrow indicates the di fference of weights on the left and the right sides of it. Is it possible to determine the weights of all stones in 4 weighings, if the balance can make a mistake in 1 gram in at most one weighing?

3   In his triangle ABC Serge made some measurements and informed Ilias about the lengths of median AD and side AC. Based on these data Ilias proved the assertion: angle CAB is obtuse,

while angle DAB is acute. Determine a ratio AD/AC and prove Ilias' assertion (for any triangle with such a ratio).

4 Baron Munchausen claims that he got a map of a country that consists of five cities. Each two cities are connected by a direct road. Each road intersects no more than one another road (and no more than once). On the map, the roads are colored in yellow or red, and while circling any city (along its border) one can notice that the colors of crossed roads alternate. Can Baron's claim be true?

5 Let $a_1, a_2, \cdots, a_n$ be a sequence of positive numbers, so that $a_1 + a_2 + \cdots + a_n \le \frac{1}{2}$. Prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) < 2.$$

Remark. I think this problem was posted before, but I can't find the link now.

6 Let ABC be a non-isosceles triangle. Two isosceles triangles $AB'C$ with base AC and $CA'B$ with base BC are constructed outside of triangle ABC. Both triangles have the same base angle φ. Let C_1 be a point of intersection of the perpendicular from C to $A'B'$ and the perpendicular bisector of the segment AB. Determine the value of $\angle AC_1B$.

7 In an in finite sequence $a_1, a_2, a_3, \cdots$, the number a_1 equals 1, and each $a_n, n > 1$, is obtained from a_{n-1} as follows:

- if the greatest odd divisor of n has residue 1 modulo 4, then $a_n = a_{n-1} + 1$,

- and if this residue equals 3, then $a_n = a_{n-1} - 1$.

Prove that in this sequence

(a) the number 1 occurs infi nitely many times;

(b) each positive integer occurs infi nitely many times.

(The initial terms of this sequence are $1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, \cdots$)

– Senior O-Level

1 Alex distributes some cookies into several boxes and records the number of cookies in each box. If the same number appears more than once, it is recorded only once. Serge takes one cookie from each box and puts them on the first plate. Then he takes one cookie from each box that is still non-empty and puts the cookies on the second plate. He continues until all the boxes are empty. Then Serge records the number of cookies on each plate. Again, if the same number appears more than once, it is recorded only once. Prove that Alex's record contains the same number of numbers as Serge's record.

2   Solve the system of equations $(n > 2)$

$$\sqrt{x_1} + \sqrt{x_2 + x_3 + \cdots + x_n} = \sqrt{x_2} + \sqrt{x_3 + x_4 + \cdots + x_n + x_1} = \cdots = \sqrt{x_n} + \sqrt{x_1 + x_2 + \cdots + x_{n-1}} \ ,$$

$$x_1 - x_2 = 1.$$

3   A 30-gon $A_1 A_2 \cdots A_{30}$ is inscribed in a circle of radius 2. Prove that one can choose a point B_k on the arc $A_k A_{k+1}$ for $1 \le k \le 29$ and a point B_{30} on the arc $A_{30} A_1$, such that the numerical value of the area of the 60-gon $A_1 B_1 A_2 B_2 \ldots A_{30} B_{30}$ is equal to the numerical value of the perimeter of the original 30-gon.

5   On the infinite chessboard several rectangular pieces are placed whose sides run along the grid lines. Each two have no squares in common, and each consists of an odd number of squares. Prove that these pieces can be painted in four colours such that two pieces painted in the same colour do not share any boundary points.

4   Five distinct positive integers form an arithmetic progression. Can their product be equal to a^{2008} for some positive integer a ?

–   Senior A-Level

1   A square board is divided by lines parallel to the board sides (7 lines in each direction, not necessarily equidistant) into 64 rectangles. Rectangles are colored into white and black in alternating order. Assume that for any pair of white and black rectangles the ratio between area of white rectangle and area of black rectangle does not exceed 2. Determine the maximal ratio between area of white and black part of the board. White (black) part of the board is the total sum of area of all white (black) rectangles.

2   Space is dissected into congruent cubes. Is it necessarily true that for each cube there exists another cube so that both cubes have a whole face in common?

3   There are N piles each consisting of a single nut. Two players in turns play the following game. At each move, a player combines two piles that contain coprime numbers of nuts into a new pile. A player who can not make a move, loses. For every $N > 2$ determine which of the players, the first or the second, has a winning strategy.

4   Let $ABCD$ be a non-isosceles trapezoid. De fine a point $A1$ as intersection of circumcircle of triangle BCD and line AC. (Choose A_1 distinct from C). Points B_1, C_1, D_1 are de fined in similar way. Prove that $A_1 B_1 C_1 D_1$ is a trapezoid as well.

5 In an in finite sequence $a_1, a_2, a_3, \cdots$, the number a_1 equals 1, and each $a_n, n > 1$, is obtained from a_{n-1} as follows:

- if the greatest odd divisor of n has residue 1 modulo 4, then $a_n = a_{n-1} + 1$,

- and if this residue equals 3, then $a_n = a_{n-1} - 1$.

Prove that in this sequence

(a) the number 1 occurs infi nitely many times;

(b) each positive integer occurs infi nitely many times.

(The initial terms of this sequence are $1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, \cdots$)

6 Let $P(x)$ be a polynomial with real coefficients so that equation $P(m) + P(n) = 0$ has infi nitely many pairs of integer solutions (m, n). Prove that graph of $y = P(x)$ has a center of symmetry.

7 A test consists of 30 true or false questions. After the test (answering all 30 questions), Victor gets his score: the number of correct answers. Victor is allowed to take the test (the same questions) several times. Can Victor work out a strategy that insure him to get a perfect score after

(a) 30th attempt?

(b) 25th attempt?

(Initially, Victor does not know any answer)

– Spring 2008

– Spring 2008

– Junior O-Level

1 In the convex hexagon $ABCDEF$, AB, BC and CD are respectively parallel to DE, EF and FA. If $AB = DE$, prove that $BC = EF$ and $CD = FA$.

2 There are ten congruent segments on a plane. Each intersection point divides every segment passing through it in the ratio $3 : 4$. Find the maximum number of intersection points.

3 There are ten cards with the number a on each, ten with b and ten with c, where a, b and c are distinct real numbers. For every five cards, it is possible to add another five cards so that the sum of the numbers on these ten cards is 0. Prove that one of a, b and c is 0.

4 Find all positive integers n such that $(n + 1)!$ is divisible by $1! + 2! + ... + n!$.

5   Each cell of a 10×10 board is painted red, blue or white, with exactly twenty of them red. No two adjacent cells are painted in the same colour. A domino consists of two adjacent cells, and it is said to be good if one cell is blue and the other is white.
(a) Prove that it is always possible to cut out 30 good dominoes from such a board.
(b) Give an example of such a board from which it is possible to cut out 40 good dominoes.
(c) Give an example of such a board from which it is not possible to cut out more than 30 good dominoes.

–   Junior A-Level

1   An integer N is the product of two consecutive integers.
(a) Prove that we can add two digits to the right of this number and obtain a perfect square.
(b) Prove that this can be done in only one way if $N > 12$

2   A line parallel to the side AC of triangle ABC cuts the side AB at K and the side BC at M. O is the intersection point of AM and CK. If $AK = AO$ and $KM = MC$, prove that $AM = KB$.

3   Alice and Brian are playing a game on a $1 \times (N + 2)$ board. To start the game, Alice places a checker on any of the N interior squares. In each move, Brian chooses a positive integer n. Alice must move the checker to the n-th square on the left or the right of its current position. If the checker moves off the board, Alice wins. If it lands on either of the end squares, Brian wins. If it lands on another interior square, the game proceeds to the next move. For which values of N does Brian have a strategy which allows him to win the game in a finite number of moves?

4   Given are finitely many points in the plane, no three on a line. They are painted in four colours, with at least one point of each colour. Prove that there exist three triangles, distinct but not necessarily disjoint, such that the three vertices of each triangle have different colours, and none of them contains a coloured point in its interior.

5   Standing in a circle are 99 girls, each with a candy. In each move, each girl gives her candy to either neighbour. If a girl receives two candies in the same move, she eats one of them. What is the minimum number of moves after which only one candy remains?

6   Do there exist positive integers a, b, c and d such that

$$\begin{cases} \dfrac{a}{b} + \dfrac{c}{d} = 1 \\ \dfrac{a}{d} + \dfrac{c}{b} = 2008 \end{cases}$$

?

7   A convex quadrilateral $ABCD$ has no parallel sides. The angles between the diagonal AC and the four sides are $55^o, 55^o, 19^o$ and 16° in some order. Determine all possible values of the acute angle between AC and BD.

–   Senior O-Level

1   same as Junior O p3

2   Can it happen that the least common multiple of $1, 2, ..., n$ is 2008 times the least common multiple of $1, 2, ..., m$ for some positive integers m and n ?

3   In triangle $ABC, \angle A = 90^o$. M is the midpoint of BC and H is the foot of the altitude from A to BC. The line passing through M and perpendicular to AC meets the circumcircle of triangle AMC again at P. If BP intersects AH at K, prove that $AK = KH$.

4   No matter how two copies of a convex polygon are placed inside a square, they always have a common point. Prove that no matter how three copies of the same polygon are placed inside this square, they also have a common point.

5   We may permute the rows and the columns of the table below.
        How may different tables can we generate?

        1 2 3 4 5 6 7
        7 1 2 3 4 5 6
        6 7 1 2 3 4 5
        5 6 7 1 2 3 4
        4 5 6 7 1 2 3
        3 4 5 6 7 1 2
        2 3 4 5 6 7 1

–   Senior A-Level

1   A triangle has an angle of measure θ. It is dissected into several triangles. Is it possible that all angles of the resulting triangles are less than θ, if
        (a) $\theta = 70^o$?
        (b) $\theta = 80^o$?

2   Alice and Brian are playing a game on the real line. To start the game, Alice places a checker on a number x where $0 < x < 1$. In each move, Brian chooses a positive number d. Alice must move the checker to either $x + d$ or $x - d$. If it lands on 0 or 1, Brian wins. Otherwise the game proceeds to the next move. For which values of x does Brian have a strategy which allows him to win the game in a finite number of moves?

3   A polynomial $x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-2} x^2 + a_{n-1} x + a_n$ has n distinct real roots $x_1, x_2, ..., x_n$, where $n > 1$. The polynomial $nx^{n-1} + (n-1)a_1 x^{n-2} + (n-2)a_2 x^{n-3} + ... + 2a_{n-2} x + a_{n-1}$ has roots $y_1, y_2, ..., y_{n_1}$.
        Prove that $\frac{x_1^2 + x_2^2 + ... + x_n^2}{n} > \frac{y_1^2 + y_2^2 + ... + y_{n-1}^2}{n-1}$

4   Each of Peter and Basil draws a convex quadrilateral with no parallel sides. The angles between a diagonal and the four sides of Peter's quadrilateral are α, α, β and γ in some order. The angles between a diagonal and the four sides of Basil's quadrilateral are also α, α, β and γ in some order. Prove that the acute angle between the diagonals of Peter's quadrilateral is equal to the acute angle between the diagonals of Basil's quadrilateral.

5   The positive integers are arranged in a row in some order, each occuring exactly once. Does there always exist an adjacent block of at least two numbers somewhere in this row such that the sum of the numbers in the block is a prime number?

6   Seated in a circle are 11 wizards. A different positive integer not exceeding 1000 is pasted onto the forehead of each. A wizard can see the numbers of the other 10, but not his own. Simultaneously, each wizard puts up either his left hand or his right hand. Then each declares the number on his forehead at the same time. Is there a strategy on which the wizards can agree beforehand, which allows each of them to make the correct declaration?

7   Each of three lines cuts chords of equal lengths in two given circles. The points of intersection of these lines form a triangle. Prove that its circumcircle passes through the midpoint of the segment joining the centres of the circles.