

**Tournament Of Towns 2010**

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– Spring - Junior O-Level

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**1** Each of six fruit baskets contains pears, plums and apples. The number of plums in each basket equals the total number of apples in all other baskets combined while the number of apples in each basket equals the total number of pears in all other baskets combined. Prove that the total number of fruits is a multiple of 31.

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**2** Karlson and Smidge divide a cake in a shape of a square in the following way. First, Karlson places a candle on the cake (chooses some interior point). Then Smidge makes a straight cut from the candle to the boundary in the direction of his choice. Then Karlson makes a straight cut from the candle to the boundary in the direction perpendicular to Smidge's cut. As a result, the cake is split into two pieces; Smidge gets the smaller one. Smidge wants to get a piece which is no less than a quarter of the cake. Can Karlson prevent Smidge from getting the piece of that size?

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**3** An angle is given in a plane. Using only a compass, one must find out (a) if this angle is acute. Find the minimal number of circles one must draw to be sure. (b) if this angle equals  $31^\circ$ . (One may draw as many circles as one needs).

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**4** At the math contest each participant met at least 3 pals who he/she already knew. Prove that the Jury can choose an even number of participants (more than two) and arrange them around a table so that each participant be set between these who he/she knows.

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**5** 101 numbers are written on a blackboard:  $1^2, 2^2, 3^2, \dots, 101^2$ . Alex choses any two numbers and replaces them by their positive difference. He repeats this operation until one number is left on the blackboard. Determine the smallest possible value of this number.

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**1** Alex has a piece of cheese. He chooses a positive number  $a \neq 1$  and cuts the piece into several pieces one by one. Every time he chooses a piece and cuts it in the same ratio  $1 : a$ . His goal is to divide the cheese into two piles of equal masses. Can he do it?

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**2** Let  $M$  be the midpoint of side  $AC$  of the triangle  $ABC$ . Let  $P$  be a point on the side  $BC$ . If  $O$  is the point of intersection of  $AP$  and  $BM$  and  $BO = BP$ , determine the ratio  $\frac{OM}{PC}$ .

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**3** Each of 999 numbers placed in a circular way is either 1 or  $-1$ . (Both values appear). Consider

the total sum of the products of every 10 consecutive numbers. (a) Find the minimal possible value of this sum. (b) Find the maximal possible value of this sum.

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4 Can it happen that the sum of digits of some positive integer  $n$  equals 100 while the sum of digits of number  $n^3$  equals  $100^3$ ?

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5  $N$  horsemen are riding in the same direction along a circular road. Their speeds are constant and pairwise distinct. There is a single point on the road where the horsemen can surpass one another. Can they ride in this fashion for arbitrarily long time? Consider the cases: (a)  $N = 3$ ; (b)  $N = 10$ .

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6 A broken line consists of 31 segments. It has no self intersections, and its start and end points are distinct. All segments are extended to become straight lines. Find the least possible number of straight lines.

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7 Several fleas sit on the squares of a  $10 \times 10$  chessboard (at most one flea per square). Every minute, all fleas simultaneously jump to adjacent squares. Each flea begins jumping in one of four directions (up, down, left, right), and keeps jumping in this direction while it is possible; otherwise, it reverses direction on the opposite. It happened that during one hour, no two fleas ever occupied the same square. Find the maximal possible number of fleas on the board.

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1 2010 ships deliver bananas, lemons and pineapples from South America to Russia. The total number of bananas on each ship equals the number of lemons on all other ships combined, while the total number of lemons on each ship equals the total number of pineapples on all other ships combined. Prove that the total number of fruits is a multiple of 31.

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2 Let  $f(x)$  be a function such that every straight line has the same number of intersection points with the graph  $y = f(x)$  and with the graph  $y = x^2$ . Prove that  $f(x) = x^2$ .

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3 Is it possible to cover the surface of a regular octahedron by several regular hexagons without gaps and overlaps? (A regular octahedron has 6 vertices, each face is an equilateral triangle, each vertex belongs to 4 faces.)

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4 Assume that  $P(x)$  is a polynomial with integer non negative coefficients, different from constant. Baron Munchausen claims that he can restore  $P(x)$  provided he knows the values of  $P(2)$  and  $P(P(2))$  only. Is the baron's claim valid?

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5 A needle (a segment) lies on a plane. One can rotate it  $45^\circ$  round any of its endpoints. Is it possible that after several rotations the needle returns to initial position with the endpoints interchanged?

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1 Is it possible to split all straight lines in a plane into the pairs of perpendicular lines, so that every line belongs to a single pair?

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2 Alex has a piece of cheese. He chooses a positive number  $a$  and cuts the piece into several pieces one by one. Every time he chooses a piece and cuts it in the same ratio  $1 : a$ . His goal is to divide the cheese into two piles of equal masses. Can he do it if  $(a)a$  is irrational?  $(b)a$  is rational,  $a \neq 1$ ?

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3 Consider a composition of functions  $\sin, \cos, \tan, \cot, \arcsin, \arccos, \arctan, \operatorname{arccot}$ , applied to the number 1. Each function may be applied arbitrarily many times and in any order. (ex:  $\sin \cos \arcsin \cos \sin \dots 1$ ). Can one obtain the number 2010 in this way?

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4 5000 movie fans gathered at a convention. Each participant had watched at least one movie. The participants should be split into discussion groups of two kinds. In each group of the first kind, the members would discuss a movie they all watched. In each group of the second kind, each member would tell about the movie that no one else in this group had watched. Prove that the chairman can always split the participants into exactly 100 groups. (A group consisting of one person is allowed; in this case this person submits a report).

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5 33 horsemen are riding in the same direction along a circular road. Their speeds are constant and pairwise distinct. There is a single point on the road where the horsemen can surpass one another. Can they ride in this fashion for arbitrarily long time ?

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6 Quadrilateral  $ABCD$  is circumscribed around the circle with centre  $I$ . Let points  $M$  and  $N$  be the midpoints of sides  $AB$  and  $CD$  respectively and let  $\frac{IM}{AB} = \frac{IN}{CD}$ . Prove that  $ABCD$  is either a trapezoid or a parallelogram.

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7 A multi-digit number is written on the blackboard. Susan puts in a number of plus signs between some pairs of adjacent digits. The addition is performed and the process is repeated with the sum. Prove that regardless of what number was initially on the blackboard, Susan can always obtain a single-digit number in at most ten steps.

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1 In a multiplication table, the entry in the  $i$ -th row and the  $j$ -th column is the product  $ij$ . From an  $m \times n$  subtable with both  $m$  and  $n$  odd, the interior  $(m-2)(n-2)$  rectangle is removed, leaving behind a frame of width 1. The squares of the frame are painted alternately black and white. Prove that the sum of the numbers in the black squares is equal to the sum of the numbers in the white squares.

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- 2 In a quadrilateral  $ABCD$  with an incircle,  $AB = CD$ ;  $BC < AD$  and  $BC$  is parallel to  $AD$ . Prove that the bisector of  $\angle C$  bisects the area of  $ABCD$ .
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- 3 A  $1 \times 1 \times 1$  cube is placed on an  $8 \times 8$  chessboard so that its bottom face coincides with a square of the chessboard. The cube rolls over a bottom edge so that the adjacent face now lands on the chessboard. In this way, the cube rolls around the chessboard, landing on each square at least once. Is it possible that a particular face of the cube never lands on the chessboard?
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- 4 In a school, more than 90% of the students know both English and German, and more than 90% percent of the students know both English and French. Prove that more than 90% percent of the students who know both German and French also know English.
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- 5 A circle is divided by  $2N$  points into  $2N$  arcs of length 1. These points are joined in pairs to form  $N$  chords. Each chord divides the circle into two arcs, the length of each being an even integer. Prove that  $N$  is even.
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- 1 A round coin may be used to construct a circle passing through one or two given points on the plane. Given a line on the plane, show how to use this coin to construct two points such that they define a line perpendicular to the given line. Note that the coin may not be used to construct a circle tangent to the given line.
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- 2 Pete has an instrument which can locate the midpoint of a line segment, and also the point which divides the line segment into two segments whose lengths are in a ratio of  $n : (n + 1)$ , where  $n$  is any positive integer. Pete claims that with this instrument, he can locate the point which divides a line segment into two segments whose lengths are at any given rational ratio. Is Pete right?
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- 3 At a circular track, 10 cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least 25 meetings.
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- 4 A rectangle is divided into  $2 \times 1$  and  $1 \times 2$  dominoes. In each domino, a diagonal is drawn, and no two diagonals have common endpoints. Prove that exactly two corners of the rectangle are endpoints of these diagonals.
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- 5 For each side of a given pentagon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.
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- 6 In acute triangle  $ABC$ , an arbitrary point  $P$  is chosen on altitude  $AH$ . Points  $E$  and  $F$  are the midpoints of sides  $CA$  and  $AB$  respectively. The perpendiculars from  $E$  to  $CP$  and from  $F$  to

$BP$  meet at point  $K$ . Prove that  $KB = KC$ .

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- 7 Merlin summons the  $n$  knights of Camelot for a conference. Each day, he assigns them to the  $n$  seats at the Round Table. From the second day on, any two neighbours may interchange their seats if they were not neighbours on the first day. The knights try to sit in some cyclic order which has already occurred before on an earlier day. If they succeed, then the conference comes to an end when the day is over. What is the maximum number of days for which Merlin can guarantee that the conference will last?

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- 1 The exchange rate in a Funny-Money machine is  $s$  McLoonies for a Loonie or  $\frac{1}{s}$  Loonies for a McLoonie, where  $s$  is a positive real number. The number of coins returned is rounded off to the nearest integer. If it is exactly in between two integers, then it is rounded up to the greater integer. (a) Is it possible to achieve a one-time gain by changing some Loonies into McLoonies and changing all the McLoonies back to Loonies? (b) Assuming that the answer to (a) is "yes", is it possible to achieve multiple gains by repeating this procedure, changing all the coins in hand and back again each time?

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- 2 The diagonals of a convex quadrilateral  $ABCD$  are perpendicular to each other and intersect at the point  $O$ . The sum of the inradii of triangles  $AOB$  and  $COD$  is equal to the sum of the inradii of triangles  $BOC$  and  $DOA$ . (a) Prove that  $ABCD$  has an incircle. (b) Prove that  $ABCD$  is symmetric about one of its diagonals.

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- 3 From a police station situated on a straight road in nite in both directions, a thief has stolen a police car. Its maximal speed equals 90

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- 4 A square board is dissected into  $n^2$  rectangular cells by  $n - 1$  horizontal and  $n - 1$  vertical lines. The cells are painted alternately black and white in a chessboard pattern. One diagonal consists of  $n$  black cells which are squares. Prove that the total area of all black cells is not less than the total area of all white cells.

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- 5 In a tournament with 55 participants, one match is played at a time, with the loser dropping out. In each match, the numbers of wins so far of the two participants differ by not more than 1. What is the maximal number of matches for the winner of the tournament?

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- 1 There are 100 points on the plane. All 4950 pairwise distances between two points have been recorded. (a) A single record has been erased. Is it always possible to restore it using the remaining records? (b) Suppose no three points are on a line, and  $k$  records were erased. What is the maximum value of  $k$  such that restoration of all the erased records is always possible?
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- 2 At a circular track,  $2n$  cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least  $n^2$  meetings.
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- 3 For each side of a given polygon, divide its length by the total length of all other sides. Prove that the sum of all the fractions obtained is less than 2.
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- 4 Two dueling wizards are at an altitude of 100 above the sea. They cast spells in turn, and each spell is of the form "decrease the altitude by  $a$  for me and by  $b$  for my rival" where  $a$  and  $b$  are real numbers such that  $0 < a < b$ . Different spells have different values for  $a$  and  $b$ . The set of spells is the same for both wizards, the spells may be cast in any order, and the same spell may be cast many times. A wizard wins if after some spell, he is still above water but his rival is not. Does there exist a set of spells such that the second wizard has a guaranteed win, if the number of spells is (a) finite; (b) infinite?
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- 5 The quadrilateral  $ABCD$  is inscribed in a circle with center  $O$ . The diagonals  $AC$  and  $BD$  do not pass through  $O$ . If the circumcentre of triangle  $AOC$  lies on the line  $BD$ , prove that the circumcentre of triangle  $BOD$  lies on the line  $AC$ .
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- 6 Each cell of a  $1000 \times 1000$  table contains 0 or 1. Prove that one can either cut out 990 rows so that at least one 1 remains in each column, or cut out 990 columns so that at least one 0 remains in each row.
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- 7 A square is divided into congruent rectangles with sides of integer lengths. A rectangle is important if it has at least one point in common with a given diagonal of the square. Prove that this diagonal bisects the total area of the important rectangles
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