JBMO ShortLists 2002

www.artofproblemsolving.com/community/c3732

by Bugi

1 A student is playing computer. Computer shows randomly 2002 positive numbers. Game's rules let do the following operations
- to take 2 numbers from these, to double first one, to add the second one and to save the sum.
- to take another 2 numbers from the remainder numbers, to double the first one, to add the second one, to multiply this sum with previous and to save the result.
- to repeat this procedure, until all the 2002 numbers won't be used.
Student wins the game if final product is maximum possible.
Find the winning strategy and prove it.

---

2 Positive real numbers are arranged in the form: $1\quad 3\quad 6\quad 10\quad 15...2\quad 5\quad 9\quad 14...4\quad 8\quad 13...7\quad 12...11...$
Find the number of the line and column where the number 2002 stays.

---

3 Let a,b,c be positive real numbers such that $abc=\frac{9}{4}$. Prove the inequality: $a^3+b^3+c^3>a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}$

Jury's variant:
Prove the same, but with $abc=2$

---

5 Let a,b,c be positive real numbers. Prove the inequality: $\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\geq\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$

---

6 Let $a_1,a_2,...,a_6$ be real numbers such that: $a_1\neq 0, a_1a_6+a_3+a_4=2a_2a_5 \text{ and } a_1a_3\geq a_2^2$
Prove that $a_4a_6\leq a_5^2$. When does equality holds?

---

7 Consider integers $a_i, i=\overline{1,2002}$ such that $a_1^{-3}+a_2^{-3}+\ldots+a_{2002}^{-3}=\frac{1}{2}$
Prove that at least 3 of the numbers are equal.

---

8 Let ABC be a triangle with centroid G and A_1,B_1,C_1 midpoints of the sides BC,CA,AB. A paralel through A_1 to BB_1 intersects B_1C_1 at F. Prove that triangles ABC and FA_1A are similar if and only if quadrilateral AB_1GC_1 is cyclic.

---

9 In triangle ABC,H,I,O are orthocenter, incenter and circumcenter, respectively. CI cuts circumcircle at L. If $AB=IL$ and $AH=OH$, find angles of triangle ABC.

---

10 Let ABC be a triangle with area S and points D,E,F on the sides BC,CA,AB. Perpendiculars at points D,E,F to the BC,CA,AB cut circumcircle of the triangle ABC at points

$(D_1, D_2), (E_1, E_2), (F_1, F_2)$. Prove that: $|D_1B \cdot D_1C - D_2B \cdot D_2C| + |E_1A \cdot E_1C - E_2A \cdot E_2C| + |F_1B \cdot F_1A - F_2B \cdot F_2A| > 4S$

11 Let ABC be an isosceles triangle with $AB = AC$ and $\angle A = 20^\circ$. On the side AC consider point D such that $AD = BC$. Find $\angle BDC$.

12 Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $BC = CD$. On the sides AB, BC, CD, DA we consider points K, L, L_1, K_1 such that quadrilateral KLL_1K_1 is rectangle. Then consider rectangles $MNPQ$ inscribed in the triangle BLK, where $M \in KB, N \in BL, P, Q \in LK$ and $M_1N_1P_1Q_1$ inscribed in triangle DK_1L_1 where P_1 and Q_1 are situated on the L_1K_1, M on the DK_1 and N_1 on the DL_1. Let S, S_1, S_2, S_3 be the areas of the $ABCD, KLL_1K_1, MNPQ, M_1N_1P_1Q_1$ respectively. Find the maximum possible value of the expression: $\frac{S_1+S_2+S_3}{S}$

13 Let $A_1, A_2, ..., A_{2002}$ be arbitrary points in the plane. Prove that for every circle of radius 1 and for every rectangle inscribed in this circle, there exist 3 vertices M, N, P of the rectangle such that $MA_1 + MA_2 + \cdots + MA_{2002} + NA_1 + NA_2 + \cdots + NA_{2002} + PA_1 + PA_2 + \cdots + PA_{2002} \geq 6006.$