JBMO ShortLists 2006

www.artofproblemsolving.com/community/c3733

by Bugi, parmenides51

1 For an acute triangle ABC prove the inequality: $\sum_{cyclic} \frac{m_a^2}{-a^2+b^2+c^2} \ge \frac{9}{4}$ where m_a, m_b, m_c are lengths of corresponding medians.

---

2 Let x, y, z be positive real numbers such that $x+2y+3z=\frac{11}{12}$. Prove the inequality $6(3xy+4xz+2yz)+6x+3y+4z+72xyz \le \frac{107}{18}$.

---

3 Let $n \ge 3$ be a natural number. A set of real numbers $\{x_1, x_2, \ldots, x_n\}$ is called *summable* if $\sum_{i=1}^{n} \frac{1}{x_i} = 1$. Prove that for every $n \ge 3$ there always exists a *summable* set which consists of n elements such that the biggest element is:
a) bigger than 2^{2n-2}
b) smaller than n^2

---

4 Determine the biggest possible value of m for which the equation $2005x + 2007y = m$ has unique solution in natural numbers.

---

5 Determine all pairs (m, n) of natural numbers for which $m^2 = nk + 2$ where $k = \overline{n1}$.

EDIT. It has been discovered the correct statement is with $k = \overline{1n}$.

---

6 Prove that for every composite number $n > 4$, numbers kn divides $(n-1)!$ for every integer k such that $1 \le k \le \lfloor \sqrt{n-1} \rfloor$.

---

7 Determine all numbers $\overline{abcd}$ such that $\overline{abcd} = 11(a+b+c+d)^2$.

---

8 Prove that there do not exist natural numbers $n \ge 10$ having all digits different from zero, and such that all numbers which are obtained by permutations of its digits are perfect squares.

---

9 Let $ABCD$ be a trapezoid with $AB \parallel CD, AB > CD$ and $\angle A + \angle B = 90^\circ$. Prove that the distance between the midpoints of the bases is equal to the semidifference of the bases.

---

10 Let $ABCD$ be a trapezoid inscribed in a circle $\mathcal{C}$ with $AB \parallel CD$, $AB = 2CD$. Let $\{Q\} = AD \cap BC$ and let P be the intersection of tangents to $\mathcal{C}$ at B and D. Calculate the area of the quadrilateral $ABPQ$ in terms of the area of the triangle PDQ.

---

11 Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. Let $M \in AB$. A line through M (different from AB) cuts circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at Z, D, E, C respectively such that $D, E \in ZC$. Perpendiculars at B to the lines EB, ZB and AD respectively cut circle $\mathcal{C}_2$ in F, K and N. Prove that $KF = NC$.

---

12 Let ABC be an equilateral triangle of center O, and $M \in BC$. Let K, L be projections of M onto the sides AB and AC respectively. Prove that line OM passes through the midpoint of the segment KL.

---

13 Let A be a subset of the set $\{1, 2, \ldots, 2006\}$, consisting of 1004 elements.
Prove that there exist 3 distinct numbers $a, b, c \in A$ such that $gcd(a, b)$:
a) divides c
b) doesn't divide c

---

14 Let $n \geq 5$ be a positive integer. Prove that the set $\{1, 2, \ldots, n\}$ can be partitioned into two non-zero subsets S_n and P_n such that the sum of elements in S_n is equal to the product of elements in P_n.

---

15 Let A_1 and B_1 be internal points lying on the sides BC and AC of the triangle ABC respectively and segments AA_1 and BB_1 meet at O. The areas of the triangles AOB_1, AOB and BOA_1 are distinct prime numbers and the area of the quadrilateral A_1OB_1C is an integer. Find the least possible value of the area of the triangle ABC, and argue the existence of such a triangle.
