

JBMO ShortLists 2012

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– Algebra

- 1 Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{b} + \frac{a}{c} + \frac{c}{b} + \frac{c}{a} + \frac{b}{c} + \frac{b}{a} + 6 \geq 2\sqrt{2} \left(\sqrt{\frac{1-a}{a}} + \sqrt{\frac{1-b}{b}} + \sqrt{\frac{1-c}{c}} \right).$$

When does equality hold?

- 2 Let a, b, c be positive real numbers such that $abc = 1$. Show that :

$$\frac{1}{a^3 + bc} + \frac{1}{b^3 + ca} + \frac{1}{c^3 + ab} \leq \frac{(ab + bc + ca)^2}{6}$$

- 3 Let a, b, c be positive real numbers such that $a + b + c = a^2 + b^2 + c^2$. Prove that :

$$\frac{a^2}{a^2 + ab} + \frac{b^2}{b^2 + bc} + \frac{c^2}{c^2 + ca} \geq \frac{a + b + c}{2}$$

- 4 Solve the following equation for $x, y, z \in \mathbb{N}$:

$$\left(1 + \frac{x}{y+z}\right)^2 + \left(1 + \frac{y}{z+x}\right)^2 + \left(1 + \frac{z}{x+y}\right)^2 = \frac{27}{4}$$

- 5 Find the largest positive integer n for which the inequality

$$\frac{a+b+c}{abc+1} + \sqrt[n]{abc} \leq \frac{5}{2}$$

holds true for all $a, b, c \in [0, 1]$. Here we make the convention $\sqrt[n]{abc} = abc$.

– Geometry

- 1 Let ABC be an equilateral triangle, and P be a point on the circumcircle of the triangle but distinct from A, B and C . The lines through P and parallel to BC, CA, AB intersect the lines CA, AB, BC at M, N and Q respectively. Prove that M, N and Q are collinear.

- 2 Let ABC be an isosceles triangle with $AB = AC$. Let also ω be a circle of center K tangent to the line AC at C which intersects the segment BC again at H . Prove that $HK \perp AB$.
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- 3 Let AB and CD be chords in a circle of center O with A, B, C, D distinct, and with the lines AB and CD meeting at a right angle at point E . Let also M and N be the midpoints of AC and BD respectively. If $MN \perp OE$, prove that $AD \parallel BC$.
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- 4 Let ABC be an acute-angled triangle with circumcircle ω , and let O, H be the triangle's circumcenter and orthocenter respectively. Let also A' be the point where the angle bisector of the angle BAC meets ω . If $A'H = AH$, then find the measure of the angle BAC .
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- 5 Let the circles k_1 and k_2 intersect at two points A and B , and let t be a common tangent of k_1 and k_2 that touches k_1 and k_2 at M and N respectively. If $t \perp AM$ and $MN = 2AM$, evaluate the angle NMB .
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- 6 Let O_1 be a point in the exterior of the circle ω of center O and radius R , and let O_1N, O_1D be the tangent segments from O_1 to the circle. On the segment O_1N consider the point B such that $BN = R$. Let the line from B parallel to ON intersect the segment O_1D at C . If A is a point on the segment O_1D other than C so that $BC = BA = a$, and if the incircle of the triangle ABC has radius r , then find the area of $\triangle ABC$ in terms of a, R, r .
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- 7 Let $MNPQ$ be a square of side length 1, and A, B, C, D points on the sides MN, NP, PQ and QM respectively such that $AC \cdot BD = \frac{5}{4}$. Can the set $\{AB, BC, CD, DA\}$ be partitioned into two subsets S_1 and S_2 of two elements each, so that each one has the sum of his elements a positive integer?

– Combinatorics

- 1 Along a round table are arranged 11 cards with the names (all distinct) of the 11 members of the 16th JBMO Problem Selection Committee. The cards are arranged in a regular polygon manner. Assume that in the first meeting of the Committee none of its 11 members sits in front of the card with his name. Is it possible to rotate the table by some angle so that at the end at least two members sit in front of the card with their names?
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- 2 On a board there are n nails, each two connected by a rope. Each rope is colored in one of n given distinct colors. For each three distinct colors, there exist three nails connected with ropes of these three colors.
- a) Can n be 6?
- b) Can n be 7?
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- 3 In a circle of diameter 1 consider 65 points, no three of them collinear. Prove that there exist three among these points which are the vertices of a triangle with area less than or equal to $\frac{1}{72}$.

 – Number Theory

1 If a, b are integers and $s = a^3 + b^3 - 60ab(a + b) \geq 2012$, find the least possible value of s .

2 Do there exist prime numbers p and q such that $p^2(p^3 - 1) = q(q + 1)$?

3 Decipher the equality :

$$(\overline{VER} - \overline{IA}) = G^{RE}(\overline{GRE} + \overline{ECE})$$

assuming that the number \overline{GREECE} has a maximum value. Each letter corresponds to a unique digit from 0 to 9 and different letters correspond to different digits. It's also supposed that all the letters G, E, V and I are different from 0.

4 Determine all triples (m, n, p) satisfying :

$$n^{2p} = m^2 + n^2 + p + 1$$

where m and n are integers and p is a prime number.

5 Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

6 If a, b, c, d are integers and $A = 2(a - 2b + c)^4 + 2(b - 2c + a)^4 + 2(c - 2a + b)^4$, $B = d(d + 1)(d + 2)(d + 3) + 1$, then prove that $(\sqrt{A} + 1)^2 + B$ cannot be a perfect square.

7 Find all $a, b, c \in \mathbb{N}$ for which

$$1997^a + 15^b = 2012^c$$
