International Zhautykov Olympiad 2005

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–	Junior
Day 1	
1	The 40 unit squares of the 9 9-table (see below) are labeled. The horizontal or vertical row of 9 unit squares is good if it has more labeled unit squares than unlabeled ones. How many good (horizontal and vertical) rows totally could have the table?
2	Let m, n be integers such that $0 \leq m \leq 2n$. Then prove that the number $2^{2n+2} + 2^{m+2} + 1$ is perfect square iff $m = n$.
3	Let A be a set of $2n$ points on the plane such that no three points are collinear. Prove that for any distinct two points $a, b \in A$ there exists a line that partitions A into two subsets each containing n points and such that a, b lie on different sides of the line.
Day 2	
1	For the positive real numbers a, b, c prove the inequality $$\frac{c}{a+2b} + \frac{a}{b+2c} + \frac{b}{c+2a} \geq 1.$$
2	Let the circle $(I; r)$ be inscribed in the triangle ABC. Let D be the point of contact of this circle with BC. Let E and F be the midpoints of BC and AD, respectively. Prove that the three points I, E, F are collinear.
3	Find all prime numbers p, q less than 2005 and such that $q \mid p^2 + 4$, $p \mid q^2 + 4$.
–	Senior
Day 1	
1	Prove that the equation $x^5 + 31 = y^2$ has no integer solution.
2	Let r be a real number such that the sequence $(a_n)_{n \geq 1}$ of positive real numbers satisfies the equation $a_1 + a_2 + \cdots + a_{m+1} \leq r a_m$ for each positive integer m. Prove that $r \geq 4$.

3 Let SABC be a regular triangular pyramid. Find the set of all points $D(D! = S)$ in the space satisfing the equation $|cosASD - 2cosBSD - 2cosCSD| = 3$.

Day 2

1 For the positive real numbers a, b, c prove that

$$\frac{c}{a+2b} + \frac{d}{b+2c} + \frac{a}{c+2d} + \frac{b}{d+2a} \geq \frac{4}{3}.$$

2 The inner point X of a quadrilateral is *observable* from the side YZ if the perpendicular to the line YZ meet it in the colosed interval $[YZ]$. The inner point of a quadrilateral is a $k-$point if it is observable from the exactly k sides of the quadrilateral. Prove that if a convex quadrilateral has a 1-point then it has a $k-$point for each $k = 2, 3, 4$.

3 Find all prime numbers $p, q < 2005$ such that $q|p^2 + 8$ and $p|q^2 + 8$.