International Zhautykov Olympiad 2006

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Day 1

1 Solve in positive integers the equation

$$n = \varphi(n) + 402,$$

where $\varphi(n)$ is the number of positive integers less than n having no common prime factors with n.

2 Let ABC be a triangle and K and L be two points on (AB), (AC) such that $BK = CL$ and let $P = CK \cap BL$. Let the parallel through P to the interior angle bisector of $\angle BAC$ intersect AC in M. Prove that $CM = AB$.

3 Let $m \geq n \geq 4$ be two integers. We call a $m \times n$ board filled with 0's or 1's *good* if

1) not all the numbers on the board are 0 or 1;

2) the sum of all the numbers in 3×3 sub-boards is the same;

3) the sum of all the numbers in 4×4 sub-boards is the same.

Find all m, n such that there exists a good $m \times n$ board.

Day 2

1 In a pile you have 100 stones. A partition of the pile in k piles is *good* if:

1) the small piles have different numbers of stones;

2) for any partition of one of the small piles in 2 smaller piles, among the $k+1$ piles you get 2 with the same number of stones (any pile has at least 1 stone).

Find the maximum and minimal values of k for which this is possible.

2 Let a, b, c, d be real numbers with sum 0. Prove the inequality:

$$(ab + ac + ad + bc + bd + cd)^2 + 12 \geq 6(abc + abd + acd + bcd).$$

3 Let $ABCDEF$ be a convex hexagon such that $AD = BC + EF$, $BE = AF + CD$, $CF = DE + AB$. Prove that:

$$\frac{AB}{DE} = \frac{CD}{AF} = \frac{EF}{BC}.$$