International Zhautykov Olympiad 2007

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Day 1

1 There are given 111 coins and a $n\times n$ table divided into unit cells. This coins are placed inside the unit cells (one unit cell may contain one coin, many coins, or may be empty), such that the difference between the number of coins from two neighbouring cells (that have a common edge) is 1. Find the maximal n for this to be possible.

2 Let $ABCD$ be a convex quadrilateral, with $\angle BAC=\angle DAC$ and M a point inside such that $\angle MBA=\angle MCD$ and $\angle MBC=\angle MDC$. Show that the angle $\angle ADC$ is equal to $\angle BMC$ or $\angle AMB$.

3 Show that there are an infinity of positive integers n such that $2^{n}+3^{n}$ is divisible by n^{2}.

Day 2

1 Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+f(y))=f(x)+\sin y$, for all reals x,y ?

2 The set of positive nonzero real numbers are partitioned into three mutually disjoint non-empty subsets $(A\cup B\cup C)$.
a) show that there exists a triangle of side-lengths a,b,c, such that $a\in A,b\in B,c\in C$.
b) does it always happen that there exists a right triangle with the above property ?

3 Let $ABCDEF$ be a convex hexagon and it's diagonals have one common point M. It is known that the circumcenters of triangles MAB,MBC,MCD,MDE,MEF,MFA lie on a circle. Show that the quadrilaterals $ABDE,BCEF,CDFA$ have equal areas.