International Zhautykov Olympiad 2008

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– Senior

Day 1

1 Points K,L,M,N are repectively the midpoints of sides AB,BC,CD,DA in a convex quadrilateral $ABCD$.Line KM meets dioganals AC and BD at points P and Q,respectively.Line LN meets dioganals AC and BD at points R and S,respectively.
Prove that if $AP\cdot PC=BQ\cdot QD$,then $AR\cdot RC=BS\cdot SD$.

2 A polynomial $P(x)$ with integer coefficients is called good,if it can be represented as a sum of cubes of several polynomials (in variable x) with integer coefficients.For example,the polynomials x^3-1 and $9x^3-3x^2+3x+7=(x-1)^3+(2x)^3+2^3$ are good.
a)Is the polynomial $P(x)=3x+3x^7$ good?
b)Is the polynomial $P(x)=3x+3x^7+3x^{2008}$ good?
Justify your answers.

3 Let $A=\{(a_1,\dots,a_8)|a_i\in\mathbb{N}$, $1\leq a_i\leq i+1$ for each $i=1,2\dots,8\}$.A subset $X\subset A$ is called sparse if for each two distinct elements $(a_1,\dots,a_8)$,$(b_1,\dots,b_8)\in X$,there exist at least three indices i,such that $a_i\neq b_i$.
Find the maximal possible number of elements in a sparse subset of set A.

Day 2

1 For each positive integer n,denote by $S(n)$ the sum of all digits in decimal representation of n.
Find all positive integers n,such that $n=2S(n)^3+8$.

2 Let A_1A_2 be the external tangent line to the nonintersecting cirlces $\omega_1(O_1)$ and $\omega_2(O_2)$,$A_1\in\omega_1$,$A_2\in\omega_2$.Points K is the midpoint of A_1A_2.And KB_1 and KB_2 are tangent lines to ω_1 and ω_2,respectvely($B_1\neq A_1$,$B_2\neq A_2$).Lines A_1B_1 and A_2B_2 meet in point L,and lines KL and O_1O_2 meet in point P.
Prove that points B_1,B_2,P and L are concyclic.

3 Let a,b,c be positive integers for which $abc=1$. Prove that $\sum\frac{1}{b(a+b)}\geq\frac{3}{2}$.

– Grade level 2