International Zhautykov Olympiad 2009

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by Erken

Day 1

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1 Find all pairs of integers (x, y), such that

$$x^2 - 2009y + 2y^2 = 0$$

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2 Find all real a, such that there exist a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following inequality:

$$x + af(y) \leq y + f(f(x))$$

for all $x, y \in \mathbb{R}$

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3 For a convex hexagon $ABCDEF$ with an area S, prove that:

$$AC \cdot (BD + BF - DF) + CE \cdot (BD + DF - BF) + AE \cdot (BF + DF - BD) \geq 2\sqrt{3}S$$

Day 2

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1 On the plane, a Cartesian coordinate system is chosen. Given points A_1, A_2, A_3, A_4 on the parabola $y = x^2$, and points B_1, B_2, B_3, B_4 on the parabola $y = 2009x^2$. Points A_1, A_2, A_3, A_4 are concyclic, and points A_i and B_i have equal abscissas for each $i = 1, 2, 3, 4$.
Prove that points B_1, B_2, B_3, B_4 are also concyclic.

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2 Given a quadrilateral $ABCD$ with $\angle B = \angle D = 90^\circ$. Point M is chosen on segment AB so taht $AD = AM$. Rays DM and CB intersect at point N. Points H and K are feet of perpendiculars from points D and C to lines AC and AN, respectively.
Prove that $\angle MHN = \angle MCK$.

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3 In a checked 17×17 table, n squares are colored in black. We call a line any of rows, columns, or any of two diagonals of the table. In one step, if at least 6 of the squares in some line are black, then one can paint all the squares of this line in black.
Find the minimal value of n such that for some initial arrangement of n black squares one can paint all squares of the table in black in some steps.
