International Zhautykov Olympiad 2011
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Day 1 January 16th

1 Given is trapezoid $ABCD$, M and N being the midpoints of the bases of AD and BC, respectively.
a) Prove that the trapezoid is isosceles if it is known that the intersection point of perpendicular bisectors of the lateral sides belongs to the segment MN.
b) Does the statement of point a) remain true if it is only known that the intersection point of perpendicular bisectors of the lateral sides belongs to the line MN?

2 Find all functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy the equality,

$$f(x+f(y))=f(x-f(y))+4xf(y)$$

for any $x,y\in\mathbb{R}$.

3 Let $\mathbb{N}$ denote the set of all positive integers. An ordered pair $(a;b)$ of numbers $a,b\in\mathbb{N}$ is called *interesting*, if for any $n\in\mathbb{N}$ there exists $k\in\mathbb{N}$ such that the number a^k+b is divisible by 2^n. Find all *interesting* ordered pairs of numbers.

Day 2 January 17th

1 Find the maximum number of sets which simultaneously satisfy the following conditions:

i) any of the sets consists of 4 elements,

ii) any two different sets have exactly 2 common elements,

iii) no two elements are common to all the sets.

2 Let n be integer, $n>1$. An element of the set $M=\{1,2,3,\ldots,n^2-1\}$ is called *good* if there exists some element b of M such that $ab-b$ is divisible by n^2. Furthermore, an element a is called *very good* if a^2-a is divisible by n^2. Let g denote the number of *good* elements in M and v denote the number of *very good* elements in M. Prove that

$$v^2+v\leq g\leq n^2-n.$$

3 Diagonals of a cyclic quadrilateral $ABCD$ intersect at point K. The midpoints of diagonals AC and BD are M and N, respectively. The circumscribed circles ADM and BCM intersect at points M and L. Prove that the points $K, L, M,$ and N lie on a circle. (all points are supposed to be different.)
