International Zhautykov Olympiad 2012

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Day 1

1 An acute triangle ABC is given. Let D be an arbitrary inner point of the side AB. Let M and N be the feet of the perpendiculars from D to BC and AC, respectively. Let H_1 and H_2 be the orthocentres of triangles MNC and MND, respectively. Prove that the area of the quadrilateral AH_1BH_2 does not depend on the position of D on AB.

2 A set of (unit) squares of a $n\times n$ table is called *convenient* if each row and each column of the table contains at least two squares belonging to the set. For each $n\geq 5$ determine the maximum m for which there exists a *convenient* set made of m squares, which becomes in*convenient* when any of its squares is removed.

3 Let P, Q, R be three polynomials with real coefficients such that

$$P(Q(x)) + P(R(x)) = \text{constant}$$

for all x. Prove that $P(x) = \text{constant}$ or $Q(x) + R(x) = \text{constant}$ for all x.

Day 2

1 Do there exist integers m, n and a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying simultaneously the following two conditions?

• i) $f(f(x)) = 2f(x) - x - 2$ for any $x \in \mathbb{R}$; • ii) $m \leq n$ and $f(m) = n$.

2 Equilateral triangles ACB' and BDC' are drawn on the diagonals of a convex quadrilateral $ABCD$ so that B and B' are on the same side of AC, and C and C' are on the same sides of BD. Find $\angle BAD + \angle CDA$ if $B'C' = AB + CD$.

3 Find all integer solutions of the equation the equation $2x^2 - y^{14} = 1$.