International Zhautykov Olympiad 2013

www.artofproblemsolving.com/community/c3743

by Amir Hossein

Day 1 January 15th

1 Given a trapezoid $ABCD$ ($AD \parallel BC$) with $\angle ABC > 90^\circ$. Point M is chosen on the lateral side AB. Let O_1 and O_2 be the circumcenters of the triangles MAD and MBC, respectively. The circumcircles of the triangles MO_1D and MO_2C meet again at the point N. Prove that the line O_1O_2 passes through the point N.

2 Find all odd positive integers $n>1$ such that there is a permutation $a_1, a_2, a_3, \ldots, a_n$ of the numbers $1, 2, 3, \ldots, n$ where n divides one of the numbers $a_k^2 - a_{k+1} - 1$ and $a_k^2 - a_{k+1} + 1$ for each k, $1 \le k \le n$ (we assume $a_{n+1}=a_1$).

3 Let a,b,c, and d be positive real numbers such that $abcd=1$. Prove that

$$\frac{(a-1)(c+1)}{1+bc+c}+\frac{(b-1)(d+1)}{1+cd+d}+\frac{(c-1)(a+1)}{1+da+a}+\frac{(d-1)(b+1)}{1+ab+b} \ge 0.$$

Proposed by Orif Ibrogimov, Uzbekistan.

Day 2 January 16th

1 A quadratic trinomial $p(x)$ with real coefficients is given. Prove that there is a positive integer n such that the equation $p(x)=\frac{1}{n}$ has no rational roots.

2 Given convex hexagon $ABCDEF$ with $AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$. The distance between the lines AB and DE is equal to the distance between the lines BC and EF and to the distance between the lines CD and FA. Prove that the sum $AD+BE+CF$ does not exceed the perimeter of hexagon $ABCDEF$.

3 A 10×10 table consists of 100 unit cells. A *block* is a 2×2 square consisting of 4 unit cells of the table. A set C of n blocks covers the table (i.e. each cell of the table is covered by some block of C) but no $n-1$ blocks of C cover the table. Find the largest possible value of n.