International Zhautykov Olympiad 2014

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by KamalDoni, wws, mikolez

Day 1

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1 Points M, N, K lie on the sides BC, CA, AB of a triangle ABC, respectively, and are different from its vertices. The triangle MNK is called *beautiful* if $\angle BAC = \angle KMN$ and $\angle ABC = \angle KNM$. If in the triangle ABC there are two beautiful triangles with a common vertex, prove that the triangle ABC is right-angled.

Proposed by Nairi M. Sedrakyan, Armenia

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2 Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:
(i) for each real y there is a real x such that $f(x) = y$, and
(ii) $f(f(x)) = (x-1)f(x) + 2$ for all real x ?

Proposed by Igor I. Voronovich, Belarus

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3 Given are 100 different positive integers. We call a pair of numbers *good* if the ratio of these numbers is either 2 or 3. What is the maximum number of good pairs that these 100 numbers can form? (A number can be used in several pairs.)

Proposed by Alexander S. Golovanov, Russia

Day 2

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1 Does there exist a polynomial $P(x)$ with integral coefficients such that $P(1+\sqrt{3}) = 2+\sqrt{3}$ and $P(3+\sqrt{5}) = 3+\sqrt{5}$?

Proposed by Alexander S. Golovanov, Russia

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2 Let $U = \{1, 2, \dots, 2014\}$. For positive integers a, b, c we denote by $f(a,b,c)$ the number of ordered 6-tuples of sets $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ satisfying the following conditions:
(i) $Y_1 \subseteq X_1 \subseteq U$ and $|X_1| = a$;
(ii) $Y_2 \subseteq X_2 \subseteq U \setminus Y_1$ and $|X_2| = b$;
(iii) $Y_3 \subseteq X_3 \subseteq U \setminus (Y_1 \cup Y_2)$ and $|X_3| = c$.
Prove that $f(a,b,c)$ does not change when a, b, c are rearranged.

Proposed by Damir A. Yeliussizov, Kazakhstan

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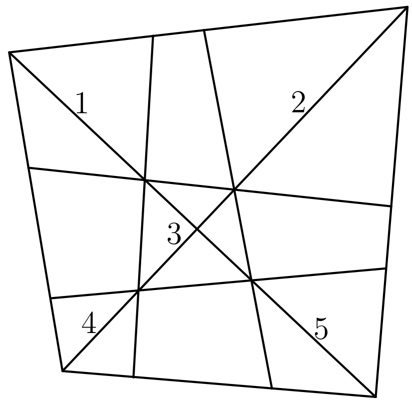
3 Four segments divide a convex quadrilateral into nine quadrilaterals. The points of intersections of these segments lie on the diagonals of the quadrilateral (see figure). It is known that the quadrilaterals 1, 2, 3, 4 admit inscribed circles. Prove that the quadrilateral 5 also has an inscribed circle.



Proposed by Nairi M. Sedrakyan, Armenia