International Zhautykov Olympiad 2015

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by Sardor, zukazuk

Day 1

1 Each point with integral coordinates in the plane is coloured white or blue. Prove that one can choose a colour so that for every positive integer n there exists a triangle of area n having its vertices of the chosen colour.

2 Inside the triangle ABC a point M is given. The line BM meets the side AC at N. The point K is symmetrical to M with respect to AC. The line BK meets AC at P. If $\angle AMP = \angle CMN$, prove that $\angle ABP = \angle CBN$.

3 Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x^3+y^3+xy) = x^2f(x)+y^2f(y)+f(xy)$, for all $x, y \in \mathbb{R}$.

Day 2

1 Determine the maximum integer n such that for each positive integer $k \le \frac{n}{2}$ there are two positive divisors of n with difference k.

2 Let A_n be the set of partitions of the sequence $1, 2, ..., n$ into several subsequences such that every two neighbouring terms of each subsequence have different parity,and B_n the set of partitions of the sequence $1, 2, ..., n$ into several subsequences such that all the terms of each subsequence have the same parity (for example,the partition $(1, 4, 5, 8), (2, 3), (6, 9), (7)$ is an element of A_9,and the partition $(1, 3, 5), (2, 4), (6)$ is an element of B_6).
Prove that for every positive integer n the sets A_n and B_{n+1} contain the same number of elements.

3 The area of a convex pentagon $ABCDE$ is S, and the circumradii of the triangles ABC, BCD, CDE, DEA, EAB are R_1, R_2, R_3, R_4, R_5. Prove the inequality

$$R_1^4 + R_2^4 + R_3^4 + R_4^4 + R_5^4 \ge \frac{4}{5\sin^2 108^\circ} S^2.$$