Indonesia TST 2010

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–	Stage 2
–	Day 1
1	Find all functions $f: R \to R$ that satisfies $$xf(y) - yf(x) = f\left(\frac{y}{x}\right)$$ for all $x, y \in R$.
2	Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that $$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$
3	For every natural number n, define $s(n)$ as the smallest natural number so that for every natural number a relatively prime to n, this equation holds: $$a^{s(n)} \equiv 1 (mod n)$$ Find all natural numbers n such that $s(n) = 2010$
4	300 parliament members are divided into 3 chambers, each chamber consists of 100 members. For every 2 members, they either know each other or are strangers to each other.Show that no matter how they are divided into these 3 chambers, it is always possible to choose 2 members, each from different chamber such that there exist 17 members from the third chamber so that all of them knows these two members, or all of them are strangers to these two members.
–	Day 2
1	Sequence u_n is defined with $u_0 = 0, u_1 = \frac{1}{3}$ and $$\frac{2}{3}u_n = \frac{1}{2}(u_{n+1} + u_{n-1})$$

$\forall n = 1, 2, ...$
Show that $|u_n| \leq 1 \; \forall n \in \mathbb{N}$.

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2 Let a_0, a_1, a_2, ... be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \geq 2^n$ for all $n \geq 0$.

Proposed by Morteza Saghafian, Iran

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3 Two parallel lines r, s and two points $P \in r$ and $Q \in s$ are given in a plane. Consider all pairs of circles (C_P, C_Q) in that plane such that C_P touches r at P and C_Q touches s at Q and which touch each other externally at some point T. Find the locus of T.

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4 Given $3n$ cards, each of them will be written with a number from the following sequence:

$$2, 3, ..., n, n+1, n+3, n+4, ..., 2n+1, 2n+2, 2n+4, ..., 3n+3$$

with each number used exactly once. Then every card is arranged from left to right in random order. Determine the probability such that for every i with $1 \leq i \leq 3n$, the number written on the i-th card, counted from the left, is greater than or equal to i.

– Day 3

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1 Given a, b, c positive real numbers satisfying $a+b+c=1$.
Prove that

$$\frac{1}{\sqrt{ab+bc+ca}} \geq \sqrt{\frac{2a}{3(b+c)}} + \sqrt{\frac{2b}{3(c+a)}} + \sqrt{\frac{2c}{3(a+b)}} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

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2 Find maximal numbers of planes, such there are 6 points and
1) 4 or more points lies on every plane.
2) No one line passes through 4 points.

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3 Given acute triangle ABC with circumcenter O and the center of nine-point circle N. Point N_1 are given such that $\angle NAB = \angle N_1AC$ and $\angle NBC = \angle N_1BA$. Perpendicular bisector of segment OA intersects the line BC at A_1. Analogously define B_1 and C_1. Show that all three points A_1, B_1, C_1 are collinear at a line that is perpendicular to ON_1.

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4 How many natural numbers (a, b, n) with $gcd(a, b) = 1$ and $n > 1$ such that the equation

$$x^{an} + y^{bn} = 2^{2010}$$

has natural numbers solution (x, y)

– Day 4

1 find all pairs of relatively prime natural numbers (m, n) in such a way that there exists non constant polynomial f satisfying

$$gcd(a + b + 1, mf(a) + nf(b) > 1$$

for every natural numbers a and b

2 Let T be a tree with$n$ vertices. Choose a positive integer k where $1 \leq k \leq n$ such that S_k is a subset with k elements from the vertices in T. For all $S \in S_k$, define $c(S)$ to be the number of component of graph from S if we erase all vertices and edges in T, except all vertices and edges in S. Determine $\sum_{S \in S_k} c(S)$, expressed in terms of n and k.

3 Given a non-isosceles triangle ABC with incircle k with center S. k touches the side BC, CA, AB at P, Q, R respectively. The line QR and line BC intersect at M. A circle which passes through B and C touches k at N. The circumcircle of triangle MNP intersects AP at L. Prove that S, L, M are collinear.

4 Given a positive integer n and $I = \{1, 2, ..., k\}$ with k is a positive integer.
Given positive integers $a_1, a_2, ..., a_k$ such that for all $i \in I$: $1 \leq a_i \leq n$ and

$$\sum_{i=1}^{k} a_i \geq 2(n!).$$

Show that there exists $J \subseteq I$ such that

$$n! + 1 \geq \sum_{j \in J} a_j > \sqrt{n! + (n-1)n}$$

– Day 5

1 Find all triplets of real numbers (x, y, z) that satisfies the system of equations $x^5 = 2y^3 + y - 2$ $y^5 = 2z^3 + z - 2$ $z^5 = 2x^3 + x - 2$

2 A government's land with dimensions $n \times n$ are going to be sold in phases. The land is divided into n^2 squares with dimension 1×1. In the first phase, n farmers bought a square, and for each rows and columns there is only one square that is bought by a farmer. After one season, each farmer could buy one more square, with the conditions that the newly-bought square has a common side with the farmer's land and it hasn't been bought by other farmers. Determine all values of n such that the government's land could not be entirely sold within n seasons.

3 Let $ABCD$ be a convex quadrilateral with AB is not parallel to CD. Circle ω_1 with center O_1 passes through A and B, and touches segment CD at P. Circle ω_2 with center O_2 passes through C and D, and touches segment AB at Q. Let E and F be the intersection of circles ω_1 and ω_2. Prove that EF bisects segment PQ if and only if BC is parallel to AD.

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4 Let n be a positive integer with $n = p^{2010}q^{2010}$ for two odd primes p and q. Show that there exist exactly $\sqrt[2010]{n}$ positive integers $x \le n$ such that $p^{2010}|x^p - 1$ and $q^{2010}|x^q - 1$.

– Stage 1

– Day 1

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1 Let a, b, and c be non-negative real numbers and let x, y, and z be positive real numbers such that $a + b + c = x + y + z$. Prove that

$$\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \ge a + b + c.$$

Hery Susanto, Malang

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2 Let $A = \{n : 1 \le n \le 2009^{2009}, n \in \mathbb{N}\}$ and let $S = \{n : n \in A, \gcd\left(n, 2009^{2009}\right) = 1\}$. Let P be the product of all elements of S. Prove that

$$P \equiv 1 \pmod{2009^{2009}}.$$

Nanang Susyanto, Jogjakarta

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3 In a party, each person knew exactly 22 other persons. For each two persons X and Y, if X and Y knew each other, there is no other person who knew both of them, and if X and Y did not know each other, there are exactly 6 persons who knew both of them. Assume that X knew Y iff Y knew X. How many people did attend the party?
Yudi Satria, Jakarta

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4 Let ABC be a non-obtuse triangle with CH and CM are the altitude and median, respectively. The angle bisector of $\angle BAC$ intersects CH and CM at P and Q, respectively. Assume that

$$\angle ABP = \angle PBQ = \angle QBC,$$

(a) prove that ABC is a right-angled triangle, and
(b) calculate $\dfrac{BP}{CH}$.
Soewono, Bandung

– Day 2

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1 Let $ABCD$ be a trapezoid such that $AB \parallel CD$ and assume that there are points E on the line outside the segment BC and F on the segment AD such that $\angle DAE = \angle CBF$. Let I, J, K respectively be the intersection of line EF and line CD, the intersection of line EF and line AB, and the midpoint of segment EF. Prove that K is on the circumcircle of triangle CDJ if and only if I is on the circumcircle of triangle ABK.
Utari Wijayanti, Bandung

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2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)$$

for all real numbers x and y.
Hery Susanto, Malang

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3 Let x, y, and z be integers satisfying the equation

$$\frac{2008}{41y^2} = \frac{2z}{2009} + \frac{2007}{2x^2}.$$

Determine the greatest value that z can take.
Budi Surodjo, Jogjakarta

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4 For each positive integer n, define $f(n)$ as the number of digits 0 in its decimal representation. For example, $f(2) = 0$, $f(2009) = 2$, etc. Please, calculate

$$S = \sum_{k=1}^{n} 2^{f(k)},$$

for $n = 9,999,999,999$.
Yudi Satria, Jakarta

– Day 3

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1 Let f be a polynomial with integer coefficients. Assume that there exists integers a and b such that $f(a) = 41$ and $f(b) = 49$. Prove that there exists an integer c such that 2009 divides $f(c)$.
Nanang Susyanto, Jogjakarta

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2 Given an equilateral triangle, all points on its sides are colored in one of two given colors. Prove that the is a right-angled triangle such that its three vertices are in the same color and on the sides of the equilateral triangle.
Alhaji Akbar, Jakarta

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3 Let $a_1, a_2, \ldots$ be sequence of real numbers such that $a_1 = 1$, $a_2 = \frac{4}{3}$, and

$$a_{n+1} = \sqrt{1 + a_n a_{n-1}}, \quad \forall n \geq 2.$$

Prove that for all $n \geq 2$,

$$a_n^2 > a_{n-1}^2 + \frac{1}{2}$$

and

$$1 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} > 2a_n.$$

Fajar Yuliawan, Bandung

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4 Let ABC be an acute-angled triangle such that there exist points D, E, F on side BC, CA, AB, respectively such that the inradii of triangle AEF, BDF, CDE are all equal to r_0. If the inradii of triangle DEF and ABC are r and R, respectively, prove that

$$r + r_0 = R.$$

Soewono, Bandung

– Day 4

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1 The integers $1, 2, \ldots, 20$ are written on the blackboard. Consider the following operation as one step: [i]choose two integers a and b such that $a - b \geq 2$ and replace them with $a - 1$ and $b + 1$[/i]. Please, determine the maximum number of steps that can be done.
Yudi Satria, Jakarta

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2 Circles Γ_1 and Γ_2 are internally tangent to circle Γ at P and Q, respectively. Let P_1 and Q_1 are on Γ_1 and Γ_2 respectively such that P_1Q_1 is the common tangent of P_1 and Q_1. Assume that Γ_1 and Γ_2 intersect at R and R_1. Define O_1, O_2, O_3 as the intersection of PQ and P_1Q_1, the intersection of PR and P_1R_1, and the intersection QR and Q_1R_1. Prove that the points O_1, O_2, O_3 are collinear.
Rudi Adha Prihandoko, Bandung

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3 Determine all real numbers a such that there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$x + f(y) = af(y + f(x))$$

for all real numbers x and y.
Hery Susanto, Malang

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4 Prove that for all integers m and n, the inequality

$$\frac{\phi(\gcd(2^m + 1, 2^n + 1))}{\gcd(\phi(2^m + 1), \phi(2^n + 1))} \geq \frac{2\gcd(m, n)}{2^{\gcd(m,n)}}$$

holds.
Nanang Susyanto, Jogjakarta

– Day 5

1 Is there a triangle with angles in ratio of $1:2:4$ and the length of its sides are integers with at least one of them is a prime number?
Nanang Susyanto, Jogjakarta

2 Consider a polynomial with coefficients of real numbers $\phi(x) = ax^3 + bx^2 + cx + d$ with three positive real roots. Assume that $\phi(0) < 0$, prove that

$$2b^3 + 9a^2d - 7abc \le 0.$$

Hery Susanto, Malang

3 Let $\mathbb{Z}$ be the set of all integers. Define the set $\mathbb{H}$ as follows:
(1). $\frac{1}{2} \in \mathbb{H}$,
(2). if $x \in \mathbb{H}$, then $\frac{1}{1+x} \in \mathbb{H}$ and also $\frac{x}{1+x} \in \mathbb{H}$.
Prove that there exists a bijective function $f : \mathbb{Z} \to \mathbb{H}$.

4 Prove that the number $(\underbrace{9999\dots99}_{2005})^{2009}$ can be obtained by erasing some digits of $(\underbrace{9999\dots99}_{2008})^{2009}$ (both in decimal representation).
Yudi Satria, Jakarta