Federal Competition For Advanced Students, Part 2 1997

www.artofproblemsolving.com/community/c3752

by Amir Hossein

Day 1

1 Let a be a fixed integer. Find all integer solutions x, y, z of the system

$$5x + (a + 2)y + (a + 2)z = a,$$

$$(2a + 4)x + (a^2 + 3)y + (2a + 2)z = 3a - 1,$$

$$(2a + 4)x + (2a + 2)y + (a^2 + 3)z = a + 1.$$

2 A positive integer K is given. Define the sequence (a_n) by $a_1 = 1$ and a_n is the n-th positive integer greater than a_{n-1} which is congruent to n modulo K.

(a) Find an explicit formula for a_n.

(b) What is the result if $K = 2$?

3 Let be given a triangle ABC. Points P on side AC and Y on the production of CB beyond B are chosen so that Y subtends equal angles with AP and PC. Similarly, Q on side BC and X on the production of AC beyond C are such that X subtends equal angles with BQ and QC. Lines YP and XB meet at R, XQ and YA meet at S, and XB and YA meet at D. Prove that $PQRS$ is a parallelogram if and only if $ACBD$ is a cyclic quadrilateral.

Day 2

1 Determine all quadruples (a, b, c, d) of real numbers satisfying the equation

$$256a^3b^3c^3d^3 = (a^6 + b^2 + c^2 + d^2)(a^2 + b^6 + c^2 + d^2)(a^2 + b^2 + c^6 + d^2)(a^2 + b^2 + c^2 + d^6).$$

2 We define the following operation which will be applied to a row of bars being situated side-by-side on positions $1, 2, \dots, N$. Each bar situated at an odd numbered position is left as is, while each bar at an even numbered position is replaced by two bars. After that, all bars will be put side-by- side in such a way that all bars form a new row and are situated on positions $1, \dots, M$. From an initial number $a_0 > 0$ of bars there originates a sequence $(a_n)_{n\geq 0}$, where an is the number of bars after having applied the operation n times.

(a) Prove that for no $n > 0$ can we have $a_n = 1997$.

(b) Determine all natural numbers that can only occur as a_0 or a_1.

3 For every natural number n, find all polynomials $x^2 + ax + b$, where $a^2 \geq 4b$, that divide $x^{2n} + ax^n + b$.