**Federal Competition For Advanced Students, Part 2 1998**

www.artofproblemsolving.com/community/c3753

by Amir Hossein

**Day 1**

**1** Let $a \geq 0$ be a natural number. Determine all rational $x$, so that

$$\sqrt{1+(a-1)\sqrt[3]{x}}=\sqrt{1+(a-1)\sqrt{x}}$$

All occurring square roots, are not negative.

**Note.** It seems the set of natural numbers = $\mathbb{N}=\{0,1,2,\ldots\}$ in this problem.

**2** Let $Q_n$ be the product of the squares of even numbers less than or equal to $n$ and $K_n$ equal to the product of cubes of odd numbers less than or equal to $n$. What is the highest power of $98$, that **a)**$Q_n$, **b)** $K_n$ or **c)** $Q_nK_n$ divides? If one divides $Q_{98}K_{98}$ by the highest power of $98$, then one get a number $N$. By which power-of-two number is $N$ still divisible?

**3** Let $a_n$ be a sequence recursively de fined by $a_0=0, a_1=1$ and $a_{n+2}=a_{n+1}+a_n$. Calculate the sum of $a_n\left(\frac{2}{5}\right)^n$ for all positive integers $n$. For what value of the base $b$ we get the sum $1$?

**Day 2**

**1** Let $M$ be the set of the vertices of a regular hexagon, our Olympiad symbol. How many chains $\emptyset \subset A \subset B \subset C \subset D \subset M$ of six different set, beginning with the empty set and ending with the $M$, are there?

**2** Let $P(x)=x^3-px^2+qx-r$ be a cubic polynomial with integer roots $a,b,c$.

**(a)** Show that the greatest common divisor of $p,q,r$ is equal to $1$ if the greatest common divisor of $a,b,c$ is equal to $1$.

**(b)** What are the roots of polynomial $Q(x)=x^3-98x^2+98sx-98t$ with $s,t$ positive integers.

**3** In a parallelogram $ABCD$ with the side ratio $AB:BC=2:\sqrt{3}$ the normal through $D$ to $AC$ and the normal through $C$ to $AB$ intersects in the point $E$ on the line $AB$. What is the relationship between the lengths of the diagonals $AC$ and $BD$?