Federal Competition For Advanced Students, Part 2 1999

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by Amir Hossein

Day 1

1 Prove that for each positive integer n, the sum of the numbers of digits of 4^n and of 25^n (in the decimal system) is odd.

2 Let ϵ be a plane and k_1, k_2, k_3 be spheres on the same side of ϵ. The spheres k_1, k_2, k_3 touch the plane at points T_1, T_2, T_3, respectively, and k_2 touches k_1 at S_1 and k_3 at S_3. Prove that the lines S_1T_1 and S_3T_3 intersect on the sphere k_2. Describe the locus of the intersection point.

3 Find all pairs (x, y) of real numbers such that

$$y^2 - [x]^2 = 19.99 \text{ and } x^2 + [y]^2 = 1999$$

where $f(x) = [x]$ is the floor function.

Day 2

1 Ninety-nine points are given on one of the diagonals of a unit square. Prove that there is at most one vertex of the square such that the average squared distance from a given point to the vertex is less than or equal to $1/2$.

2 Given a real number A and an integer n with $2 \le n \le 19$, find all polynomials $P(x)$ with real coefficients such that $P(P(P(x))) = Ax^n + 19x + 99$.

3 Two players A and B play the following game. An even number of cells are placed on a circle. A begins and A and B play alternately, where each move consists of choosing a free cell and writing either O or M in it. The player after whose move the word OMO (OMO = *Osterreichische Mathematik Olympiade*) occurs for the first time in three successive cells wins the game. If no such word occurs, then the game is a draw. Prove that if player B plays correctly, then player A cannot win.