Federal Competition For Advanced Students, Part 2 2000

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Day 1

1 The sequence an is defined by $a_0 = 4, a_1 = 1$ and the recurrence formula $a_{n+1} = a_n + 6a_{n-1}$. The sequence b_n is given by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

Find the coefficients α, β so that b_n satisfies the recurrence formula $b_{n+1} = \alpha b_n + \beta b_{n-1}$. Find the explicit form of b_n.

2 A trapezoid $ABCD$ with $AB \parallel CD$ is inscribed in a circle k. Points P and Q are chose on the arc $ADCB$ in the order $A-P-Q-B$. Lines CP and AQ meet at X, and lines BP and DQ meet at Y. Show that points P, Q, X, Y lie on a circle.

3 Find all real solutions to the equation

$$||||||x^2 - x - 1| - 3| - 5| - 7| - 9| - 11| - 13| = x^2 - 2x - 48.$$

Day 2

1 In a non-equilateral acute-angled triangle ABC with $\angle C = 60^\circ$, U is the circumcenter, H the orthocenter and D the intersection of AH and BC. Prove that the Euler line HU bisects the angle BHD.

2 Find all pairs of integers (m, n) such that

$$\left|(m^2 + 2000m + 999999) - (3n^3 + 9n^2 + 27n)\right| = 1.$$

3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x, y, z it holds that

$$f(x + f(y + z)) + f(f(x + y) + z) = 2y.$$