Federal Competition For Advanced Students, Part 2 2001

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by Amir Hossein

Day 1

1 Prove that $\frac{1}{25}\sum_{k=0}^{2001}\left[\frac{2^k}{25}\right]$ is a positive integer.

2 Determine all triples of positive real numbers (x,y,z) such that

$$x+y+z=6,$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2-\frac{4}{xyz}.$$

3 A triangle ABC is inscribed in a circle with center U and radius r. A tangent c' to a larger circle $K(U,2r)$ is drawn so that C lies between the lines $c=AB$ and C'. Lines a' and b' are analogously defined. The triangle formed by a',b',c' is denoted $A'B'C'$. Prove that the three lines, joining the midpoints of pairs of parallel sides of the two triangles, have a common point.

Day 2

1 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x,y

$$f(f(x)^2+f(y))=xf(x)+y.$$

2 Determine all integers m for which all solutions of the equation $3x^3-3x^2+m=0$ are rational.

3 Let be given a semicircle with the diameter AB, and points C,D on it such that $AC=CD$. The tangent at C intersects the line BD at E. The line AE intersects the arc of the semicircle at F. Prove that $CF<FD$.