Federal Competition For Advanced Students, Part 2 2003

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Day 1

1 Consider the polynomial $P(n) = n^3 - n^2 - 5n + 2$. Determine all integers n for which $P(n)^2$ is a square of a prime.

I'm not sure if the statement of this problem is correct, because if $P(n)^2$ be a square of a prime, then $P(n)$ should be that prime, and I don't think the problem means that.

2 Let a, b, c be nonzero real numbers for which there exist $\alpha, \beta, \gamma \in \{-1, 1\}$ with $\alpha a + \beta b + \gamma c = 0$. What is the smallest possible value of

$$\left(\frac{a^3 + b^3 + c^3}{abc}\right)^2 ?$$

3 For every lattice point (x, y) with x, y non-negative integers, a square of side $\frac{0.9}{2^x 5^y}$ with center at the point (x, y) is constructed. Compute the area of the union of all these squares.

Day 2

1 Prove that, for any integer $g > 2$, there is a unique three-digit number $\overline{abc}_g$ in base g whose representation in some base $h = g \pm 1$ is $\overline{cba}_h$.

2 We are given sufficiently many stones of the forms of a rectangle 2×1 and square 1×1. Let $n > 3$ be a natural number. In how many ways can one tile a rectangle $3 \times n$ using these stones, so that no two 2×1 rectangles have a common point, and each of them has the longer side parallel to the shorter side of the big rectangle?

3 Let ABC be an acute-angled triangle. The circle k with diameter AB intersects AC and BC again at P and Q, respectively. The tangents to k at A and Q meet at R, and the tangents at B and P meet at S. Show that C lies on the line RS.