Federal Competition For Advanced Students, Part 2 2006

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by FelixD

Day 1

1 Let N be a positive integer. How many non-negative integers $n \leq N$ are there that have an integer multiple, that only uses the digits 2 and 6 in decimal representation?

2 Let a, b, c be positive real numbers. Show that $3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$.

3 The triangle ABC is given. On the extension of the side AB we construct the point R with $BR = BC$, where $AR > BR$ and on the extension of the side AC we construct the point S with $CS = CB$, where $AS > CS$. Let A_1 be the point of intersection of the diagonals of the quadrilateral $BRSC$.
Analogous we construct the point T on the extension of the side BC, where $CT = CA$ and $BT > CT$ and on the extension of the side BA we construct the point U with $AU = AC$, where $BU > AU$. Let B_1 be the point of intersection of the diagonals of the quadrilateral $CTUA$.
Likewise we construct the point V on the extension of the side CA, where $AV = AB$ and $CV > AV$ and on the extension of the side CB we construct the point W with $BW = BA$ and $CW > BW$. Let C_1 be the point of intersection of the diagonals of the quadrilateral $AVWB$.
Show that the area of the hexagon $AC_1BA_1CB_1$ is equal to the sum of the areas of the triangles ABC and $A_1B_1C_1$.

Day 2

1 For which rational x is the number $1 + 105 \cdot 2^x$ the square of a rational number?

2 Find all monotonous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following functional equation:

$$f(f(x)) = f(-f(x)) = f(x)^2.$$

3 Let A be an integer not equal to 0. Solve the following system of equations in $\mathbb{Z}^3$. $x+y^2+z^3 = A$ $\frac{1}{x} + \frac{1}{y^2} + \frac{1}{z^3} = \frac{1}{A}$ $xy^2z^3 = A^2$

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