Federal Competition For Advanced Students, Part 2 2007

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Day 1

1 For which non-negative integers $a < 2007$ the congruence $x^2+a \equiv 0 \mod 2007$ has got exactly two different non-negative integer solutions?
That means, that there exist exactly two different non-negative integers u and v less than 2007, such that $u^2 + a$ and $v^2 + a$ are both divisible by 2007.

2 Find all tuples $(x_1, x_2, x_3, x_4, x_5, x_6)$ of non-negative integers, such that the following system of equations holds: $x_1x_2(1 - x_3) = x_4x_5$
$x_2x_3(1 - x_4) = x_5x_6$
$x_3x_4(1 - x_5) = x_6x_1$
$x_4x_5(1 - x_6) = x_1x_2$
$x_5x_6(1 - x_1) = x_2x_3$
$x_6x_1(1 - x_2) = x_3x_4$

3 Determine all rhombuses $ABCD$ with the given length $2a$ of ist sides by giving the angle $\alpha = \angle BAD$, such that there exists a circle which cuts each side of the rhombus in a chord of length a.

Day 2

1 Let M be the set of all polynomials $P(x)$ with pairwise distinct integer roots, integer coefficients and all absolut values of the coefficients less than 2007. Which is the highest degree among all the polynomials of the set M?

2 38th Austrian Mathematical Olympiad 2007, round 3 problem 5

Given is a convex n-gon with a triangulation, that is a partition into triangles through diagonals that dont cut each other. Show that its always possible to mark the n corners with the digits of the number 2007 such that every quadrilateral consisting of 2 neighbored (along an edge) triangles has got 9 as the sum of the numbers on its 4 corners.

3 The triangle ABC with the circumcircle $k(U, r)$ is given. On the extension of the radii UA a point P is chosen. The reflection of the line PB on the line BA is called g. Likewise the reflection of the line PC on the line CA is called h. The intersection of g and h is called Q.
Find the geometric location of all possible intersections Q, while P passes through the extension of the radii UA.

