Federal Competition For Advanced Students, Part 2 2008

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by April

Day 1

1 Prove the inequality

$$\sqrt{a^{1-a}b^{1-b}c^{1-c}} \le \frac{1}{3}$$

holds for all positive real numbers a, b and c with $a+b+c=1$.

2 (a) Does there exist a polynomial $P(x)$ with coefficients in integers, such that $P(d) = \frac{2008}{d}$ holds for all positive divisors of 2008?
(b) For which positive integers n does a polynomial $P(x)$ with coefficients in integers exists, such that $P(d) = \frac{n}{d}$ holds for all positive divisors of n?

3 We are given a line g with four successive points P, Q, R, S, reading from left to right. Describe a straightedge and compass construction yielding a square $ABCD$ such that P lies on the line AD, Q on the line BC, R on the line AB and S on the line CD.

Day 2

1 Determine all functions f mapping the set of positive integers to the set of non-negative integers satisfying the following conditions:
(1) $f(mn) = f(m) + f(n)$,
(2) $f(2008) = 0$, and
(3) $f(n) = 0$ for all $n \equiv 39 \pmod{2008}$.

2 Which positive integers are missing in the sequence $\{a_n\}$, with $a_n = n + [\sqrt{n}] + [\sqrt[3]{n}]$ for all $n \ge 1$? ($[x]$ denotes the largest integer less than or equal to x, i.e. g with $g \le x < g+1$.)

3 We are given a square $ABCD$. Let P be a point not equal to a corner of the square or to its center M. For any such P, we let E denote the common point of the lines PD and AC, if such a point exists. Furthermore, we let F denote the common point of the lines PC and BD, if such a point exists. All such points P, for which E and F exist are called acceptable points. Determine the set of all acceptable points, for which the line EF is parallel to AD.

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