Federal Competition For Advanced Students, Part 2 2011

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Day 1

1 Every brick has 5 holes in a line. The holes can be filled with bolts (fi tting in one hole) and braces (fi tting into two neighboring holes). No hole may remain free.
One puts n of these bricks in a line to form a pattern from left to right. In this line no two braces and no three bolts may be adjacent.
How many diff erent such patterns can be produced with n bricks?

2 We consider permutations f of the set $\mathbb{N}$ of non-negative integers, i.e. bijective maps f from $\mathbb{N}$ to $\mathbb{N}$, with the following additional properties:

$$f(f(x)) = x \quad \text{and} \quad |f(x) - x| \leqslant 3 \quad \text{for all } x \in \mathbb{N}.$$

Further, for all integers $n > 42$,

$$M(n) = \frac{1}{n+1} \sum_{j=0}^{n} |f(j) - j| < 2,011.$$

Show that there are infinitely many natural numbers K such that f maps the set

$$\{n \mid 0 \leqslant n \leqslant K\}$$

onto itself.

3 We are given a non-isosceles triangle ABC with incenter I. Show that the circumcircle k of the triangle AIB does not touch the lines CA and CB.
Let P be the second point of intersection of k with CA and let Q be the second point of intersection of k with CB.
Show that the four points A, B, P and Q (not necessarily in this order) are the vertices of a trapezoid.

Day 2

1 Determine all pairs (a, b) of non-negative integers, such that $a^b + b$ divides $a^{2b} + 2b$.
(Remark: $0^0 = 1$.)

2 Let k and n be positive integers.
Show that if x_j $(1 \leqslant j \leqslant n)$ are real numbers with $\sum_{j=1}^{n} \frac{1}{x_j^{2^k}+k} = \frac{1}{k}$, then

$$\sum_{j=1}^{n} \frac{1}{x_j^{2^{k+1}}+k+2} \leqslant \frac{1}{k+1}.$$

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3 Two circles k_1 and k_2 with radii r_1 and r_2 touch each outside at point Q. The other endpoints of the diameters through Q are P on k_1 and R on k_2.
We choose two points A and B, one on each of the arcs PQ of k_1. ($PBQA$ is a convex quadrangle.)
Further, let C be the second point of intersection of the line AQ with k_2 and let D be the second point of intersection of the line BQ with k_2.
The lines PB and RC intersect in U and the lines PA and RD intersect in V .
Show that there is a point Z that lies on all of these lines UV.
