Federal Competition For Advanced Students, Part 2 2012

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Day 1

1 Determine the maximum value of m, such that the inequality

$$(a^2+4(b^2+c^2))(b^2+4(a^2+c^2))(c^2+4(a^2+b^2)) \ge m$$

holds for every $a,b,c \in \mathbb{R} \setminus \{0\}$ with $\left|\frac{1}{a}\right|+\left|\frac{1}{b}\right|+\left|\frac{1}{c}\right| \le 3$.
When does equality occur?

2 Solve over $\mathbb{Z}$:

$$x^4y^3(y-x)=x^3y^4-216$$

3 We call an isosceles trapezoid $PQRS$ *interesting*, if it is inscribed in the unit square $ABCD$ in such a way, that on every side of the square lies exactly one vertex of the trapezoid and that the lines connecting the midpoints of two adjacent sides of the trapezoid are parallel to the sides of the square.

Find all interesting isosceles trapezoids and their areas.

Day 2

1 Given a sequence $< a_1, a_2, a_3, \cdots >$ of real numbers, we define m_n as the arithmetic mean of the numbers a_1 to a_n for $n \in \mathbb{Z}^+$.
If there is a real number C, such that

$$(i-j)m_k+(j-k)m_i+(k-i)m_j=C$$

for every triple (i,j,k) of distinct positive integers, prove that the sequence $< a_1, a_2, a_3, \cdots >$ is an arithmetic progression.

2 We define N as the set of natural numbers $n < 10^6$ with the following property:

There exists an integer exponent k with $1 \le k \le 43$, such that $2012 | n^k - 1$.

Find $|N|$.

3 Given an equilateral triangle ABC with sidelength 2, we consider all equilateral triangles PQR with sidelength 1 such that

-P lies on the side AB,
-Q lies on the side AC, and
-R lies in the inside or on the perimeter of ABC.

Find the locus of the centroids of all such triangles PQR.
