Federal Competition For Advanced Students, Part 2 2013

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Day 1

1 For each pair (a,b) of positive integers, determine all non-negative integers n such that

$$b+\left\lfloor\frac{n}{a}\right\rfloor=\left\lceil\frac{n+b}{a}\right\rceil.$$

2 Let k be an integer. Determine all functions $f\colon \mathbb{R}\to\mathbb{R}$ with $f(0)=0$ and

$$f(x^ky^k)=xyf(x)f(y)\qquad \text{for } x,y\neq 0.$$

3 A square and an equilateral triangle are inscribed in a same circle. The seven vertices form a convex heptagon S inscribed in the circle (S might be a hexagon if two vertices coincide). For which positions of the triangle relative to the square does S have the largest and smallest area, respectively?

Day 2

4 For a positive integer n, let $a_1,a_2,\dots a_n$ be nonnegative real numbers such that for all real numbers $x_1>x_2>\ldots>x_n>0$ with $x_1+x_2+\ldots+x_n<1$, the inequality $\sum_{k=1}^n a_kx_k^3<1$ holds. Show that

$$na_1+(n-1)a_2+\ldots+(n-j+1)a_j+\ldots+a_n\leqslant\frac{n^2(n+1)^2}{4}.$$

5 Let $n\geqslant 3$ be an integer. Let $A_1A_2\dots A_n$ be a convex n-gon. Consider a line g through A_1 that does not contain a further vertice of the n-gon. Let h be the perpendicular to g through A_1. Project the n-gon orthogonally on h.
For $j=1,\dots,n$, let B_j be the image of A_j under this projection. The line g is called admissible if the points B_j are pairwise distinct.
Consider all convex n-gons and all admissible lines g. How many different orders of the points $B_1,\dots,B_n$ are possible?

6 Consider a regular octahedron $ABCDEF$ with lower vertex E, upper vertex F, middle cross-section $ABCD$, midpoint M and circumscribed sphere k. Further, let X be an arbitrary point inside the face ABF. Let the line EX intersect k in E and Z, and the plane $ABCD$ in Y. Show that $\sphericalangle EMZ = \sphericalangle EYF$.